Online Appendix to "Signaling with Private Monitoring" (not for publication)

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S.1 Section 2: Omitted Proofs

Uniqueness of equilibrium in public case: Lemma S.1 provides the closed-form solution for $r = 0$, so consider $r \in (0, \infty)$. Existence of LME has already been established, so suppose by way of contradiction that there exist multiple LMEs. Then in particular, there must exist two nonidentical pairs (β_3, γ) and (β_3', γ') solving the (backward) ODEs for the respective variables with $\beta_1 = 1 - \beta_3$ and $\beta_1' = 1 - \beta_3'$. It must be that $\gamma_0 \neq \gamma_0'$; if instead $\gamma_0 = \gamma_0'$, then since $\beta_{30} = \beta_{30}' = 1/2$, by uniqueness of solutions we would have $(\beta_3, \gamma) = (\beta_3', \gamma')$, contradicting multiplicity. Suppose without loss of generality that $\gamma_0 > \gamma_0'$. Now $\gamma_T = \gamma_T' = \gamma^o$, so let $\tau := \inf\{t \in (0, T] : \gamma_t = \gamma_t'\} \leq T$. Since γ, γ' are continuous, we also have $\tau > 0$. There must exist a positive measure set $\mathcal{T}$ of times $t \in (0, \tau)$ such that $\beta_{3t}' > \beta_{3t}$, otherwise integration of the ODEs for γ, γ' would imply that $\gamma_\tau > \gamma_\tau'$, a contradiction. But for any $t \in [0, \tau)$, if $\beta_{3t} = \beta_{3t}'$, then the fact that $\gamma_t > \gamma_t'$ implies that $\dot{\beta}_{3t} > \dot{\beta}_{3t}'$, so in $[0, \tau)$, β_3 can only cross β_3' from below. As $\beta_{30} = \beta_{30}' = 1/2$, it must be that $\beta_{3t} > \beta_{3t}'$ for all $t \in (0, \tau)$, contradicting that $\mathcal{T}$ has positive measure. We conclude that there is a unique LME. $\square$

Omitted steps of proof of Proposition 2: Imposing the first-order condition on the RHS of the HJB equation and evaluating at $a = \beta_{0t}\mu + \beta_{1t}m + \beta_{3t}\theta$ yields

$$0 = -4[\beta_{0t}\mu + \beta_{1t}m + \beta_{3t}\theta] + 2\theta + \frac{\alpha_t\gamma_t[v_{2t} + 2mv_{4t} + \theta v_{5t}]}{\sigma_Y^2} + 2[\beta_{0t}\mu + \mu\beta_{1t}(1 - \chi_t) + m\alpha_t].$$

Matching coefficients in this equation, we obtain

$$v_{2t} = \frac{2\mu\sigma_Y^2[\beta_{0t} - \beta_{1t}(1-\chi_t)]}{\alpha_t\gamma_t},\ v_{4t} = \frac{-\sigma_Y^2[\beta_{3t} - \beta_{1t}(2-\chi_t)]}{\alpha_t\gamma_t},\ v_{5t} = \frac{2\sigma_Y^2[2\beta_{3t}-1]}{\alpha_t\gamma_t}, \tag{S.1}$$

provided that $\alpha_t, \gamma_t \neq 0\ \forall t$, which we will confirm. Setting $v_{iT} = 0$ for $i = 2, 4, 5$ yields the terminal conditions stated below. As in the public benchmark, these conditions also yield the equilibrium coefficients for the case where both players behave myopically.

We evaluate the HJB at $a = \beta_{0t}\mu + \beta_{1t}m + \beta_{3t}\theta$, using (S.1) to eliminate $(v_{2t}, v_{4t}, v_{5t}, \dot{v}_{2t}, \dot{v}_{4t}, \dot{v}_{5t})$ and in turn using that $\dot{\gamma}_t = -\frac{\alpha_t^2\gamma_t^2}{\sigma_Y^2}$ and $\dot{\chi}_t = \frac{\alpha_t^2\gamma_t}{\sigma_Y^2}(1-\chi_t)$ to eliminate $\dot{\gamma}_t$ and $\dot{\chi}_t$. The resulting equation yields a system of ODEs for $(v_0, v_1, v_3, \beta_0, \beta_1, \beta_3, \gamma)$:

$$\dot{v}_{0t} = rv_{0t} + \alpha_t^2\gamma_t\chi_t \tag{S.2}$$

$$\dot{v}_{1t} = rv_{1t} - 2\mu\beta_{3t}[\beta_{0t} - \beta_{1t}(1-\chi_t)] \tag{S.3}$$

$$\dot{v}_{3t} = 1 + rv_{3t} - 2\beta_{3t}^2 \tag{S.4}$$

$$\dot{\beta}_{0t} = -\frac{\alpha_t}{2\sigma_Y^2}\left\{-r\sigma_Y^2\beta_{0t}(2-\chi_t) + r\sigma_Y^2(1-\chi_t) - 2\gamma_t\beta_{1t}^2(1-\chi_t)\right\} \tag{S.5}$$

$$\dot{\beta}_{1t} = -\frac{\alpha_t}{2\sigma_Y^2}\left\{r\sigma_Y^2 - 2\beta_{1t}[\beta_{3t}\gamma_t + r\sigma_Y^2(2-\chi_t)] + 2\beta_{1t}^2\gamma_t(1-\chi_t)\right\} \tag{S.6}$$

$$\dot{\beta}_{3t} = -\frac{\alpha_t}{2\sigma_Y^2}\left\{r\sigma_Y^2(2-\chi_t) + 2\beta_{3t}[\beta_{1t}\gamma_t - r\sigma_Y^2(2-\chi_t)]\right\} \tag{S.7}$$

$$\dot{\gamma}_t = -\frac{\alpha_t^2\gamma_t^2}{\sigma_Y^2} \tag{S.8}$$

subject to the boundary conditions $v_{0T} = v_{1T} = v_{3T} = 0$, $\beta_{0T} = \frac{1-\chi_T}{2(2-\chi_T)}$, $\beta_{1T} = \frac{1}{2(2-\chi_T)}$, $\beta_{3T} = 1/2$ and $\gamma_0 = \gamma^o$, where $\chi \equiv 1 - \gamma/\gamma^o$. Equations (S.5)-(S.8) form the boundary value problem in $(\beta_0, \beta_1, \beta_3, \gamma)$ referred to in the proof of Proposition 2. As the equations for v_0, v_1 and v_3 are uncoupled from each other and linear in themselves, they have unique solutions given any solution to the BVP. The α-ODE is

$$\dot{\alpha}_t = -r\alpha_t[1 - \alpha_t(2-\chi_t)]. \tag{S.9}$$

That $\alpha > 0$ in any solution to the BVP is shown in the proof in the main text, and that $\gamma > 0$ follows from the fact that α is finite. Hence v_2, v_4 and v_5 are well-defined by way of (S.1) given any solution to the BVP. □

S.1.1 Proofs for Comparisons between Public and No-Feedback Cases

To prove Propositions 3 and 4 we rely on two sets of results. Lemmas S.1-S.2 below obtain closed-form solutions when $r = 0$. Lemmas S.3-S.4 provide the equilibrium coefficients for the fully myopic cases and establish uniform convergence of coefficients as $r \nearrow \infty$.

Lemma S.1 (Closed-form solutions when $r = 0$)**.** *For $r = 0$, the leadership game has a unique LME for the public case, and $(\beta_0, \beta_1, \beta_3, \gamma)$ satisfy $\beta_0 \equiv 0$, $\beta_1 \equiv 1 - \beta_3$,*

$$\gamma_t = \frac{\gamma_T}{2} + \frac{1}{\frac{2}{\gamma_T} - \frac{T-t}{\sigma_Y^2}}, \beta_{3t} = \frac{1}{2 - \frac{\gamma_T(T-t)}{2\sigma_Y^2}}, \text{ and } \gamma_T = \frac{\gamma^o T + 2\sigma_Y^2 - \sqrt{(\gamma^o T)^2 + 4\sigma_Y^4}}{T}. \quad \text{(S.10)}$$

Proof. Observe that $\dot{\beta}_{3t}\gamma_t + \beta_{3t}\dot{\gamma}_t = \frac{\beta_{3t}^2\gamma_t^2}{\sigma_Y^2}$. Hence, define $\Pi_t := \beta_{3t}\gamma_t$, which has ODE $\dot{\Pi}_t = \frac{\Pi_t^2}{\sigma_Y^2}$ with initial condition $\Pi_0 = \beta_{30}\gamma^F = \gamma^F/2$; its solution is $\Pi_t = \left[\frac{2}{\gamma_{Pub}^F} - \frac{t}{\sigma_Y^2}\right]^{-1}$. Substitute Π into $\dot{\gamma}_t = -\frac{\beta_{3t}^2\gamma_t^2}{\sigma_Y^2}$ to obtain $\dot{\gamma}_t = \frac{1}{\sigma_Y^2}\left[\frac{1}{\frac{2}{\gamma_{Pub}^F} - \frac{t}{\sigma_Y^2}}\right]^2$ which implies $\gamma_t = C_\gamma + \frac{1}{\frac{2}{\gamma_{Pub}^F} - \frac{t}{\sigma_Y^2}}$. As $\gamma_0 = \gamma_{Pub}^F$, we have $C_\gamma = \gamma_{Pub}^F/2$ and thus

$$\gamma_t = \frac{\gamma_{Pub}^F}{2} + \left[\frac{2}{\gamma_{Pub}^F} - \frac{t}{\sigma_Y^2}\right]^{-1}. \quad \text{(S.11)}$$

Moreover, $\gamma_T = \gamma^o = \frac{\gamma_{Pub}^F}{2} + \left[\frac{2}{\gamma_{Pub}^F} - \frac{T}{\sigma_Y^2}\right]^{-1}$, which is equivalent to the quadratic $\frac{T}{2}\left(\gamma_{Pub}^F\right)^2 - \left(\gamma^o T + 2\sigma_Y^2\right)\gamma_{Pub}^F + 2\sigma_Y^2\gamma^o = 0$. The quadratic on the LHS is convex and evaluates to $2\sigma_Y^2\gamma^o > 0$ at $\gamma_{Pub}^F = 0$ and evaluates to $-\left(\gamma^o\right)^2 T/2 < 0$ at $\gamma_{Pub}^F = \gamma^o$, so there is a unique solution in $(0, \gamma^o)$ which in the forward system is γ_T as in the proposition statement. Substituting this into (S.11) and returning to the forward system by replacing t with $T - t$ yields γ_t in the forward system. It is easy to verify that $\gamma_t > 0$ for all t.

Lastly, we have (in the forward system) $\beta_{3t} = \Pi_t/\gamma_t = \frac{1}{2 - \frac{\gamma_T^{Pub}(T-t)}{2\sigma_Y^2}}$ and $\beta_{1t} = 1 - \beta_{3t}$. □

Lemma S.2 (Closed-form solution no-feedback case $r = 0$)**.** *For $r = 0$, the leadership game has a unique LME for the no feedback case:*

$$\beta_{1t} = \frac{\gamma^o[(\gamma^o + \gamma_T)^2\sigma_Y^2 - (T-t)(\gamma^o)^2\gamma_T]}{(\gamma^o + \gamma_T)[2\sigma_Y^2(\gamma^o + \gamma_T)^2 - (T-t)(\gamma^o)^2\gamma_T]}, \quad \beta_{3t} = \frac{\sigma_Y^2(\gamma^o + \gamma_T)^2}{2\sigma_Y^2(\gamma^o + \gamma_T)^2 - (T-t)(\gamma^o)^2\gamma_T}$$

$$\alpha_t = \frac{\gamma^o}{\gamma^o + \gamma_T}, \quad \gamma_t = \frac{\gamma_T\sigma_Y^2\left(\gamma^o + \gamma_T\right)^2}{\sigma_Y^2\left(\gamma^o + \gamma_T\right)^2 - (T-t)\left(\gamma^o\right)^2\gamma_T},$$

for all $t \in [0, T]$, where $\chi_t = 1 - \gamma_t/\gamma^o$ and $\gamma_T \in (0, \gamma^o)$ is the unique solution in $(0, \gamma^o)$ to the cubic $q(\gamma) := \gamma T\left(\gamma^o\right)^3 + \left(\gamma - \gamma^o\right)\left(\gamma + \gamma^o\right)^2\sigma_Y^2 = 0$, and $\beta_0 \equiv 1 - \beta_1 - \beta_3$.

Proof. We work with the backward system. By reversing time, the α-ODE (S.9) becomes $\dot{\alpha}_t = r\alpha[1-\alpha_t(2-\chi_t)]$; with $r = 0$, α must be constant and equal to its initial value $\alpha_0 = \frac{1}{2-\chi_0}$. Next, recall that by Lemma A.2, $\chi_t = 1 - \frac{\gamma_t}{\gamma^o}$, so $\chi_0 = 1 - \frac{\gamma^F_{NF}}{\gamma^o}$ and thus $\alpha_t = \alpha = \frac{\gamma^o}{\gamma^F_{NF}+\gamma^o}$ for all $t \in [0, T]$. Next, note that the ODE $\dot{\gamma}_t = \frac{\alpha^2\gamma_t^2}{\sigma_Y^2}$ given an initial value γ^F_{NF} has solution $\gamma_t = \frac{\gamma^F_{NF}\sigma_Y^2}{\sigma_Y^2 - \gamma^F_{NF}\left(\frac{\gamma^o}{\gamma^F_{NF}+\gamma^o}\right)^2 t}$; switching back to the forward system by replacing t with $T-t$ yields the expression in the original statement. Now the terminal condition $\gamma_T = \gamma^o$ is equivalent to the following cubic equation for γ^F_{NF}:

$$q(\gamma^F_{NF}) := \gamma^F_{NF}T\,(\gamma^o)^3 + \left(\gamma^F_{NF} - \gamma^o\right)\left(\gamma^F_{NF} + \gamma^o\right)^2 \sigma_Y^2 = 0. \tag{S.12}$$

Note $q\left(\gamma^F_{NF}\right) > 0$ for $\gamma^F_{NF} \geq \gamma^o$ and $q\left(\gamma^F_{NF}\right) \leq 0$ for $\gamma^F_{NF} \leq 0$, so all real roots must lie in $(0, \gamma^o)$. Now any root to the cubic must satisfy

$$\frac{T(\gamma^o)^3}{\gamma^o - \gamma^F_{NF}} = \sigma_Y^2 \frac{(\gamma^F_{NF} + \gamma^o)^2}{\gamma^F_{NF}}. \tag{S.13}$$

The LHS of (S.13) is strictly increasing for $\gamma^F_{NF} \in (0, \gamma^o)$ while the RHS is strictly decreasing in this interval, so q has a unique real root. Returning to the β_1 ODE, using $\alpha = \beta_1\chi + \beta_3$, we have $\dot{\beta}_1 = \frac{\alpha\gamma_t\beta_{1t}}{\sigma_Y^2}(\alpha - \beta_{1t})$. This ODE can be solved by integration after moving $\beta_1(\alpha-\beta_1)$ to the LHS, and with algebra, one obtains (in the forward system) the expression in the proposition statement. One then obtains β_{3t} from these using $\beta_{3t} = \alpha - \beta_{1t}\chi_t$. □

Lemma S.3 (Closed-form solutions—myopic case)**.** *Suppose the leader is myopic. In the LME for the public case, $\beta_3 = 1/2$ and $\gamma_t^{Pub} = \frac{4\sigma_Y^2\gamma^o}{4\sigma_Y^2+\gamma^o t}$. In the LME for the no feedback case, $\alpha_t = \frac{\gamma^o}{\gamma^0+\gamma_t^{NF}}$, where γ_t^{NF} is defined implicitly as the unique solution in $(0, \gamma^o]$ of the equation $f(\gamma/\gamma^o) := 2\ln(\gamma_t^{NF}/\gamma^o) - \gamma^o/\gamma_t^{NF} + \gamma_t^{NF}/\gamma^o = -\frac{\gamma^o t}{\sigma_Y^2}$.*

Proof. We first consider the public benchmark case, where in the myopic solution, $\beta_{3t} = 1/2$, and thus (in the forward system) $\dot{\gamma}_t^{Pub} = -\frac{\beta_{3t}^2(\gamma_t^{Pub})^2}{\sigma_Y^2} = -\frac{(\gamma_t^{Pub})^2}{4\sigma_Y^2}$ which implies $\gamma_t^{Pub} = \frac{4\sigma_Y^2\gamma^o}{4\sigma_Y^2+\gamma^o t}$.

In the myopic solution to the no feedback case, $\alpha_t = \frac{1}{2-\chi_t} = \frac{1}{1+\gamma_t^{NF}/\gamma^o}$, where γ_t^{NF} solves the ODE

$$\dot{\gamma}_t^{NF} = -\frac{\alpha_t^2\left(\gamma_t^{NF}\right)^2}{\sigma_Y^2} = -\frac{1}{\sigma_Y^2}\left(\frac{\gamma^o\gamma_t^{NF}}{\gamma^o + \gamma_t^{NF}}\right)^2 \tag{S.14}$$

$$\implies 2\frac{\dot{\gamma}_t^{NF}}{\gamma_t^{NF}} + \gamma^o\frac{\dot{\gamma}_t^{NF}}{(\gamma_t^{NF})^2} + \frac{\dot{\gamma}_t^{NF}}{\gamma^o} = -\frac{\gamma^o}{\sigma_Y^2}. \tag{S.15}$$

By integrating both sides of (S.15) and using that $\gamma_0^{NF} = \gamma^o$ to pin down the constant of integration, we obtain that γ_t^{NF} solves

$$2\ln(\gamma_t^{NF}/\gamma^o) - \gamma^o/\gamma_t^{NF} + \gamma_t^{NF}/\gamma^o = -\frac{\gamma^o t}{\sigma_Y^2}. \tag{S.16}$$

To verify that $\gamma_t^{NF} \in (0, \gamma^0]$ is well-defined as such, define $f : (0,1] \to \mathbb{R}$ by $f(y) := 2\ln(y) - 1/y + y$, and note that $f(y)$ is strictly increasing as $f'(y) = (1+1/y)^2 > 0$, and moreover, $f(1) = 0 \geq -\frac{\gamma^o t}{\sigma_Y^2}$ while $\lim_{y\to 0} f(y) = -\infty < -\frac{\gamma^o t}{\sigma_Y^2}$. It follows that for all $t \in [0,T]$, $\gamma_t^{NF} \in (0, \gamma^o]$ is uniquely determined by (S.16). □

Lemma S.4 (Uniform Convergence as $r \to \infty$)**.** *As $r \to \infty$, the solutions to the public and no feedback cases converge uniformly to their corresponding myopic solutions.*

Proof. To make the statement precise, fix any $T > 0$ and let $\{r_n\}_{n=1}^{\infty}$ be a sequence with $\lim_{n\to\infty} r_n = \infty$. Let $\{(\beta_1^{pub,n}, \beta_3^{pub,n}, \gamma^{pub,n})\}_{n=1}^{\infty}$ and $\{(\alpha^{NF,n}, \gamma^{NF,n}, \chi^{NF,n})\}_{n=1}^{\infty}$ be sequences of solutions to the BVP indexed when $r = r_n > 0$ in the public and no-feedback cases, respectively, and let $(\beta_1^{pub,\infty}, \beta_3^{pub,\infty}, \gamma^{pub,\infty})$ and $(\alpha^{NF,\infty}, \gamma^{NF,\infty}, \chi^{NF,\infty})$ denote the equilibrium coefficients for a myopic leader. We show that uniformly, $(\beta_1^{pub,n}, \beta_3^{pub,n}, \gamma^{pub,n}) \to (\beta_1^{pub,\infty}, \beta_3^{pub,\infty}, \gamma^{pub,\infty})$ and $(\alpha^{NF,n}, \gamma^{NF,n}, \chi^{NF,n}) \to (\alpha^{NF,\infty}, \gamma^{NF,\infty}, \chi^{NF,\infty})$.

To this end, for arbitrary $T > 0$ and $r \geq 0$, let $\text{BVP}^{Pub}(r)$ denote the boundary value problem for $(\beta_1, \beta_3, \gamma)$, parameterized by r. Likewise, let $\text{BVP}^{NF}(r)$ denote the boundary value problem for (α, γ, χ) defined by (S.8)-(S.9) and the χ-ODE, which we eliminate via $\chi_t = 1 - \gamma_t/\gamma^o$, and the associated boundary conditions.

$$\Xi^{Pub} := \{(\beta_1, \beta_3, \gamma) \in C^1([0,T])^3 : \exists r \geq 0 \text{ for which } (\beta_1, \beta_3, \gamma) \text{ solves } \text{BVP}^{Pub}(r)\}$$
$$\Xi^{NF} := \{(\alpha, \gamma, \chi) \in C^1([0,T])^3 : \exists r \geq 0 \text{ for which } (\alpha, \gamma, \chi) \text{ solves } \text{BVP}^{NF}(r)\}.$$

Equicontinuity We show that the families $\{(\dot\beta_1, \dot\beta_3, \dot\gamma) : (\beta_1, \beta_3, \gamma) \in \Xi^{Pub}\}$ and $\{(\dot\alpha, \dot\gamma, \dot\chi) : (\alpha, \gamma, \chi) \in \Xi^{NF}\}$ are uniformly bounded, and hence Ξ^{Pub} and Ξ^{NF} are equicontinuous.

We begin with the public case. Recall that Ξ^{Pub} is uniformly bounded, and in particular, we have $(\beta_{1t}, \beta_{3t}, \gamma_t) \in [0, 1/2] \times [1/2, 1] \times [0, \gamma^o]$ for all $(\beta_1, \beta_3, \gamma) \in \Xi^{Pub}$ and all $t \in [0,T]$. It follows that $|\dot\gamma_t| = \frac{\beta_{3t}^2 \gamma_t^2}{\sigma_Y^2} \leq \frac{(\gamma^o)^2}{\sigma_Y^2}$. We now establish a uniform bound on $\dot\beta_3$. If we define $\beta_3^m := 1/2$ and $\beta_3^f = \beta_3 - \beta_3^m$, we have from the (backward system) β_3-ODE

$$\dot\beta_{3t}^f = \dot\beta_{3t} = \beta_{3t}[-2r\beta_{3t}^f + \beta_{3t}(1-\beta_{3t})\gamma_t/\sigma_Y^2],$$

which is linear in β_3^f. Solving this ODE and multiplying through by r, we obtain

$$r\beta_{3t}^f = \int_0^t re^{-r\int_s^t 2\beta_{3u}du}\beta_{3s}^2(1-\beta_{3s})\gamma_s/\sigma_Y^2 ds.$$

Now $|\beta_{3s}^2(1-\beta_{3s})\gamma_s/\sigma_Y^2| \le \bar{g}^{Pub} := \frac{\gamma^o}{\sigma_Y^2}$. Moreover, $2\beta_{3u} \ge 1$, and thus

$$|r\beta_{3t}^f| \le \bar{g}^{Pub} \int_0^t re^{-r\int_s^t 2\beta_{3u}du} ds \le \bar{g}^{Pub} \int_0^t re^{-r(t-s)} ds = \bar{g}^{Pub}(1-e^{-rt}) < \bar{g}^{Pub}.$$

It follows that

$$\begin{aligned} |\dot{\beta}_{3t}| = |\dot{\beta}_{3t}^f| &= |\beta_{3t}[-2r\beta_{3t}^f + \beta_{3t}(1-\beta_{3t})\gamma_t/\sigma_Y^2]| \\ &\le |-2\beta_{3t}| \cdot |r\beta_{3t}^f| + |\beta_{3t}^2(1-\beta_{3t})\gamma_t/\sigma_Y^2| \le 2 \cdot \bar{g}^{Pub} + \bar{g}^{Pub}, \end{aligned}$$

which is the desired uniform bound as $\bar{g}^{Pub}$ is independent of r. Now since $\beta_1 + \beta_3 \equiv 1$, $|\dot{\beta}_{1t}|$ is also uniformly bounded above by $3\bar{g}^{Pub}$. Hence we have established uniform bounds on the derivatives $(\dot{\beta}_1, \dot{\beta}_3, \dot{\gamma})$ for $(\beta_1, \beta_3, \gamma) \in \Xi^{Pub}$, and thus Ξ^{Pub} is equicontinuous.

Next, we turn to the no feedback case, where we recall the uniform bounds $\alpha_t \in [1/(2-\chi_t), 1] \subseteq [1/2, 1]$, $\gamma_t \in [0, \gamma^o]$ and $\chi_t \in [0,1]$. Immediately, we have $|\dot{\gamma}_t| = |\frac{\alpha_t^2\gamma_t^2}{\sigma_Y^2}| \le \frac{(\gamma^o)^2}{\sigma_Y^2}$, and since $\chi \equiv 1 - \gamma/\gamma^o$, $|\dot{\chi}_t| = |-\dot{\gamma}_t/\gamma^o| \le \frac{\gamma^o}{\sigma_Y^2} =: \bar{g}^{NF}$. We now uniformly bound $\dot{\alpha}_t$.

Set $\alpha_t^m := 1/(2-\chi_t)$ and $\alpha_t^f := \alpha_t - \alpha_t^m$, and note that $\dot{\alpha}_t^m = \dot{\chi}_t/(2-\chi_t)^2$. We have $\dot{\alpha}_t^f = -r\alpha_t(2-\chi_t)\alpha_t^f - \dot{\chi}_t/(2-\chi_t)^2$, which is linear in α^f. Solving this ODE and multiplying through by r yields

$$\begin{aligned} r\alpha_t^f &= \int_0^t re^{-r\int_s^t \alpha_u(2-\chi_u)du}[-\dot{\chi}_s/(2-\chi_s)^2] ds \\ \implies |r\alpha_t^f| &\le \int_0^t re^{-r\int_s^t \alpha_u(2-\chi_u)du} |\dot{\chi}_s/(2-\chi_s)^2| ds. \end{aligned}$$

Now $|\dot{\chi}_s/(2-\chi_s)^2| \le |\dot{\chi}_s| \le \bar{g}^{NF}$ as noted above, so

$$|r\alpha_t^f| \le \bar{g}^{NF} \int_0^t re^{-r\int_s^t \alpha_u(2-\chi_u)du} ds \le \bar{g}^{NF} \int_0^t r^{-r(t-s)} ds = \bar{g}^{NF}(1-e^{-rt}) < \bar{g}^{NF},$$

where we have used that $\alpha_u \ge 1/(2-\chi_u) \implies \int_s^t \alpha_u(2-\chi_u)du \ge (t-s)$. We now have

$$|\dot{\alpha}_t| = |-r\alpha_t(2-\chi_t)\alpha_t^f| = |r\alpha_t^f| \cdot |\alpha_t| \cdot |2-\chi_t| \le \bar{g}^{NF} \cdot 1 \cdot 2.$$

Hence $(\dot{\alpha}, \dot{\gamma}, \dot{\chi})$ are uniformly bounded for $(\alpha, \gamma, \chi) \in \Xi^{NF}$, so Ξ^{NF} is equicontinuous.

Completion of the proof of Lemma S.4: Having established equicontinuity, we now complete the proof of the lemma. First, note that $r_n \to \infty$ implies that $r_n = 0$ for at most finitely many n, so there exists N such that $r_n > 0$ for all $n \ge N$; it suffices to consider only such n. In the public benchmark, recall that $\beta_1^{pub,\infty} = \beta_3^{pub,\infty} = 1/2$. We have $\beta_3 \geq 1/2 > 0$, so the β_3-ODE can be rearranged to obtain

$$\beta_{3t}^{pub,n} = 1/2 + \frac{1}{r_n}\left[\frac{(\beta_{3t}^{pub,n})^2(1-\beta_{3t}^{pub,n})\gamma_t^{pub,n}}{2\beta_{3t}^{pub,n}\sigma_Y^2} - \frac{\dot{\beta}_{3t}^{pub,n}}{2\beta_{3t}^{pub,n}}\right].$$

Using $\beta_3 \geq 1/2$ and our bound on $|\dot{\beta}_3|$, the expression in brackets is uniformly bounded in absolute value by some constant K^{Pub} (independent of r_n and t). Hence $\beta_3^{pub,n}$ converges pointwise to $1/2 = \beta_3^{pub,\infty}$. Since $\beta_1^{pub,n} \equiv 1 - \beta_3^{pub,n}$, $\beta_1^{pub,n}$ converges pointwise to $1/2 = \beta_1^{pub,\infty}$. As $[0,T]$ is compact and the sequence $\{(\beta_1^{pub,n}, \beta_3^{pub,n})\}_{n=1}^{\infty}$ is equicontinuous, we apply Lemma 39 in Royden (1988, p. 168) and obtain uniform convergence for β_1^{Pub} and β_3^{Pub}. Finally, we prove uniform convergence for γ^{Pub}, by proving pointwise convergence and invoking the same result to obtain uniform convergence. Note that for $i \in \mathbb{N} \cup \{\infty\}$, $\gamma_t^{pub,i} = \frac{\gamma^o \sigma_Y^2}{\sigma_Y^2 + \gamma^o \int_t^T (\beta_{3s}^{pub,i})^2 ds}$. Since for $n \in \mathbb{N}$, the functions $(\beta_{3s}^{pub,n})^2$ are bounded uniformly by a constant and converge pointwise to $(\beta_3^{pub,\infty})^2$, the dominated convergence theorem (Royden, 1988, p. 267, Theorem 16) implies $\int_t^T (\beta_{3s}^{pub,n})^2 ds \to \int_t^T (\beta_{3s}^{pub,\infty})^2 ds$ and thus for all $t \in [0,T]$ pointwise, $\gamma_t^{pub,n} \to \frac{\gamma^o \sigma_Y^2}{\sigma_Y^2 + \gamma^o \int_t^T (\beta_{3s}^{pub,\infty})^2 ds} = \gamma_t^{pub,\infty}$. Since the sequence of $\gamma^{pub,n}$ is equicontinuous, Royden (1988, p. 168, Lemma 39) gives uniform convergence to $\gamma^{pub,\infty}$.

Next, consider the no feedback case. By Royden (1988, p. 169, Theorem 40), there exists a subsequence of $\{(\alpha^{NF,n}, \gamma^{NF,n}, \chi^{NF,n})\}_{n=1}^{\infty}$ indexed by $\{k(n)\}_{n=1}^{\infty}$ which converges uniformly on $[0,T]$ to some limit $(\alpha^*, \gamma^*, \chi^*)$. We argue that $(\alpha^*, \gamma^*, \chi^*) = (\alpha^{NF,\infty}, \gamma^{NF,\infty}, \chi^{NF,\infty})$, and thus the original sequence converges uniformly to $(\alpha^{NF,\infty}, \gamma^{NF,\infty}, \chi^{NF,\infty})$.

Using a similar dominated convergence argument to before, we have that pointwise $\gamma_t^{NF,k(n)} \to \frac{\gamma^o \sigma_Y^2}{\sigma_Y^2 + \gamma^o \int_t^T (\alpha^*)^2 ds}$, so $\gamma_t^* = \frac{\gamma^o \sigma_Y^2}{\sigma_Y^2 + \gamma^o \int_t^T (\alpha^*)^2 ds}$ and $\chi_t^{NF,k(n)} \to 1 - \gamma_t^*/\gamma^o$. Since $\chi^{NF,k(n)} \to \chi^*$ uniformly, we have $\chi^* = 1 - \gamma^*/\gamma^o$.

Let $\alpha_t^{m,n} := 1/(2 - \chi_t^{NF,n})$. Since $\alpha_t^{NF,n} > 0$ for all n, the α ODE can be rearranged to obtain $\alpha_t^{NF,n} - \alpha_t^{m,n} = \frac{1}{r_n}\left[-\frac{\dot{\alpha}_t^{NF,n}}{\alpha_t^{NF,n}(2-\chi_t^{NF,n})}\right]$. Since $2 - \chi_t^{NF,n} > 1$ and $\alpha_t^{NF,n} > 1/2$, the expression in brackets is uniformly bounded in absolute value by some constant K^{NF} (independent of r_n and t), and thus $|\alpha_t^{NF,n} - \alpha_t^{m,n}| \to 0$ pointwise. It follows that pointwise, and by familiar arguments uniformly, $\alpha^{m,k(n)} \to \alpha^*$. But $\alpha^{m,k(n)} = 1/(2 - \chi^{NF,n}) \to 1/(2 - \chi^*)$, so $\alpha^* = 1/(2 - \chi^*) = 1/(1 + \gamma^*/\gamma^o)$.

By differentiating the equation $\gamma_t^* = \frac{\gamma^o \sigma_Y^2}{\sigma_Y^2 + \gamma^o \int_t^T (\alpha^*)^2 ds}$, we obtain $\dot{\gamma}_t^* = \frac{(\alpha_t^* \gamma_t^*)^2}{\sigma_Y^2} = \frac{(\gamma_t^*)^2}{\sigma_Y^2 [1 + \gamma_t^*/\gamma^o]^2}$, subject to initial condition $\gamma_T^* = \gamma^o$. This equation has a unique solution which is $\gamma^{NF,\infty}$ as obtained in the proof of Lemma S.3, from which $\alpha^* = \alpha^{NF,\infty}$ and $\chi^* = \chi^{NF,\infty}$. This establishes that all convergent subsequences have the same limit, and hence the original sequences converges to that limit. □

Proof of Proposition 3

We first prove the learning comparison in (i). Recall that γ_T^{NF} is the unique positive root of the cubic equation $q(\gamma)=0$ defined in Lemma S.2. At γ_T^{NF}, it is easy to deduce that q must cross 0 from below, and hence to prove the claim, it suffices to show that $q(\gamma_T^{Pub})>0$. By direct calculation, we have

$$\begin{aligned} q(\gamma_T^{Pub}) = &+\frac{\sigma_Y^2}{T^3}\left(2\sigma_Y^2-\sqrt{(T\gamma^o)^2+4\sigma_Y^4}\right)\left(2T\gamma^o+2\sigma_Y^2-\sqrt{(T\gamma^o)^2+4\sigma_Y^4}\right)^2 \\ &+(\gamma^o)^3\left(T\gamma^o+2\sigma_Y^2-\sqrt{(T\gamma^o)^2+4\sigma_Y^4}\right)=(\gamma^o)^4Tq_2(S), \qquad \text{where} \\ q_2(S) := 1+2S-\sqrt{1+4S^2}&+S\left(2S-\sqrt{1+4S^2}\right)\left(2+2S-\sqrt{1+4S^2}\right)^2 \text{ and } S:=\frac{\sigma_Y^2}{T\gamma^o}. \end{aligned}$$

We now show that $q_2(S)>0$ for all $S>0$ (observe that $q_2(0)=0$). Let $R(S)=1+2S-\sqrt{1+4S^2}$; it is straightforward to verify that $R(0)=0$ and that for all $S\geq 0$, $R'(S)>0$ and $R(S)<1$. Moreover, the inverse of R is the function $S:[0,1)\to[0,\infty)$ characterized by $S(R):=\frac{R(2-R)}{4(1-R)}$. Hence, by change of variables, $q_2(S)>0$ for all $S>0$ iff $q_3(R)>0$, where $q_3(R):=R-S(R)(1-R)(R+1)^2$. Now for $R\in[0,1)$, $q_3(R)>0$ if and only if $S(R)=\frac{R(2-R)}{4(1-R)}<\frac{R}{(1-R)(R+1)^2}$, if and only $q_4(R):=(2-R)(R+1)^2<4$. It is straightforward to verify that over the interval $[0,1]$, $q_4(R)$ attains its maximum value of 4 at $R=1$, and tracing our steps backwards this implies that $q(\gamma_T^{Pub})>0$.

Now we establish the ranking of signaling coefficients at time zero, i.e., that $\beta_{30}^{Pub}>\alpha^{NF}$. Using the associated expressions from Lemmas S.1 and S.2, this is equivalent to

$$\frac{1}{2-\frac{\gamma_T^{Pub}T}{2\sigma_Y^2}}>\frac{\gamma^o}{\gamma^o+\gamma_T^{NF}} \iff \hat{\gamma}:=\gamma^o\left(1-\frac{\gamma_T^{Pub}T}{2\sigma_Y^2}\right)<\gamma_T^{NF}.$$

It suffices to show that $q(\hat{\gamma})=T\hat{\gamma}(\gamma^o)^3+(\hat{\gamma}-\gamma^o)(\hat{\gamma}+\gamma^o)^2\sigma_Y^2<0$. Using the expression for γ_T^{Pub} from Lemma S.1, one can show that

$$q(\hat{\gamma})=-\frac{T(\gamma^o)^4}{2\sigma_Y^4}\left[(T\gamma^o)^2+2\sigma_Y^4-T\gamma^o\sqrt{(T\gamma^o)^2+4\sigma_Y^4}\right].$$

The expression in square brackets can be written as $\frac{x+y}{2}-\sqrt{xy}>0$ where $x=(T\gamma^o)^2>0$ and $y=(T\gamma^o)^2+4\sigma_Y^4>0$, and thus $q(\hat{\gamma})<0$, concluding the proof.

Finally, to prove (ii), observe first that $\gamma_t^{Pub}>\gamma_t^{NF}$ for all $t\in(0,T]$ in the fully myopic case. Indeed, since $\gamma_0^{NF}=\gamma_0^{Pub}=\gamma^o$, solving the ODEs for γ^{Pub} and γ^{NF} by integration and using that $\alpha_t\geq\beta_{3t}=1/2$ with strict inequality for all $t>0$ delivers the result. This implies that for any $\delta \in (0, T)$, $0 < \bar{\gamma} := \min_{t\in[T-\delta,T]}(\gamma_t^{Pub,\infty} - \gamma_t^{NF,\infty})$, where we use $\gamma^{x,r}$ to denote the solution for the case $x \in \{Pub, NF\}$ and r is the discount rate. By Lemma S.4, $\gamma^{Pub,r} - \gamma^{NF,r}$ converges uniformly to $\gamma^{Pub,\infty} - \gamma^{NF,\infty}$ as $r \to \infty$, and thus for any $\epsilon \in (0, \bar{\gamma})$, there exists $\bar{r} > 0$ such that for all $r > \bar{r}$ and all $t \in [T - \delta, T]$, we have $\gamma_t^{Pub,r} - \gamma_t^{NF,r} > \gamma_t^{Pub,\infty} - \gamma_t^{NF,\infty} - \epsilon \geq \bar{\gamma} - \epsilon > 0$, as desired.

Figure 1 suggests that the learning result of Proposition 3 extends to all $r \in (0, \infty)$.

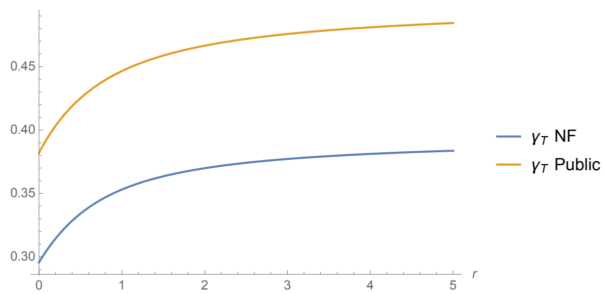

Figure 1: Terminal values of γ^{Pub} and γ^{NF}, Parameter values: $\gamma^o = \sigma_Y = 1$, and $T = 4$.

Proof of Proposition 4

We begin by calculating the leader's ex ante flow payoffs in both cases.

Lemma S.5. *The leader's ex ante flow payoffs in the public and no-feedback cases are* $\gamma_t^{Pub}[(1-\beta_{3t})^2 + \beta_{3t}^2]$ *and* $(1-\alpha_t)^2\gamma^o + \alpha_t^2\gamma_t^{NF}$, *respectively.*

Proof. Ex ante flows are given $\hat{\mathbb{E}}_0\left[(\theta - a_t)^2 + (a_t - \hat{a}_t)^2\right]$, where $\hat{\mathbb{E}}_0$ is the leader's ex ante expectation operator. In the public case, $a_t = (1 - \beta_{3t})M_t + \beta_{3t}\theta$ and $\hat{a}_t = M_t$, so we have

$$\hat{\mathbb{E}}_0\left[(\theta - a_t)^2 + (a_t - \hat{a}_t)^2\right] = \hat{\mathbb{E}}_0\left[(\theta - M_t)^2\left([1-\beta_{3t}]^2 + \beta_{3t}^2\right)\right] = \gamma_t^{Pub}[(1-\beta_{3t})^2 + \beta_{3t}^2],$$

where in the last step we have used the law of iterated expectations and the fact that the variance is deterministic: $\hat{\mathbb{E}}_0\left[(\theta - M_t)^2\right] = \hat{\mathbb{E}}_0\left[\hat{\mathbb{E}}_t[(\theta - M_t)^2]\right] = \gamma_t^{Pub}$.

In the no-feedback case, we have $a_t = (1 - \alpha_t)\mu + \alpha_t\theta$ and $\hat{a}_t = (1 - \alpha_t)\mu + \alpha_t\hat{M}_t$, so

$$\hat{\mathbb{E}}_0\left[(\theta - a_t)^2 + (a_t - \hat{a}_t)^2\right] = (1-\alpha_t)^2\hat{\mathbb{E}}_0\left[(\theta - \mu)^2\right] + \alpha_t^2\hat{\mathbb{E}}_0\left[(\theta - \hat{M}_t)^2\right] = (1-\alpha_t)^2\gamma^o + \alpha_t^2\gamma_t^{NF},$$

where we have used the definition of γ^o and $\hat{\mathbb{E}}_0\left[(\theta - \hat{M}_t)^2\right] = \hat{\mathbb{E}}_0\left[\hat{\mathbb{E}}_t[(\theta - \hat{M}_t)^2]\right] = \gamma_t^{NF}$. □

For part (ii) of the proposition, observe that from the second terms of the expressions in Lemma S.5, the undiscounted coordination losses for $i \in \{Pub, NF\}$ are $\int_0^T (x_t^i)^2\gamma_t^i\,dt$, where $x^{Pub} = \beta_3^{Pub}$ and $x^{NF} = \alpha^{NF}$. Since $\dot{\gamma}_t^i = -(x_t^i \gamma_t^i)^2/\sigma_Y^2$, such losses are $\int_0^T -\sigma_Y^2 \frac{\dot{\gamma}_t^i}{\gamma_t^i}\, dt = -\sigma_Y^2 \ln(\gamma_t^i)|_0^T = \sigma_Y^2 \ln(\gamma^o/\gamma_T^i)$.

Turning to part (i) of the proposition, for $i \in \{Pub, NF\}$, let V^i denote the long-run player's ex ante expected payoff and define $\tilde{T}^i := \frac{T\gamma_T^i}{\sigma_Y^2}$. Define $\rho := \gamma_T^{NF}/\gamma^o$. We begin by proving that for $r = 0$, $V^{Pub} = -\sigma_Y^2 \left\{\tilde{T}^{Pub}/2 - \ln\left[\frac{16-8\tilde{T}^{Pub}}{(4-\tilde{T}^{Pub})^2}\right]\right\}$ and $V^{NF} = -\sigma_Y^2 \left\{\rho(1-\rho) - \ln \rho\right\}$.

First consider the public benchmark, where from Lemma S.5,

$$V^{Pub} = \mathbb{E}_0 \left(\int_0^T \left[-(\theta - a_t)^2 - (a_t - \hat{a}_t)^2 \right] dt \right) = -\int_0^T \left[\gamma_t \left([1 - \beta_{3t}]^2 + \beta_{3t}^2 \right) \right] dt.$$

Using the closed-form expressions for γ_t and β_{3t}, the integrand simplifies to

$$\gamma_t \left([1 - \beta_{3t}]^2 + \beta_{3t}^2 \right) = \frac{\gamma_T^{Pub}}{2} \left[1 + \frac{2t\gamma_T^{Pub}\sigma_Y^2}{(2\sigma_Y^2 - t\gamma_T^{Pub})(4\sigma_Y^2 - t\gamma_T^{Pub})} \right] = \frac{\gamma_T^{Pub}}{2} \left[1 + \frac{2\tilde{t}^{Pub}}{(2 - \tilde{t}^{Pub})(4 - \tilde{t}^{Pub})} \right],$$

where $\tilde{t}^{Pub} := t\gamma_T^{Pub}/\sigma_Y^2$. Using that the function $g : x \mapsto \frac{x}{(2-x)(4-x)}$ has antiderivative $\ln\left(\frac{(4-x)^2}{2-x}\right)$ and integrating the second term w.r.t. $\tilde{t}^{Pub}$ over $\left[0, \tilde{T}^{Pub}\right]$ and rescaling by $\sigma_Y^2/\gamma_T^{Pub}$, we obtain

$$V^{Pub} = -T\frac{\gamma_T^{Pub}}{2} - \sigma_Y^2 \left(\ln\left[\frac{(4 - \tilde{T}^{Pub})^2}{2 - \tilde{T}^{Pub}}\right] - \ln 8 \right) = -\sigma_Y^2 \left\{ \tilde{T}^{Pub}/2 - \ln\left[\frac{16 - 8\tilde{T}^{Pub}}{(4 - \tilde{T}^{Pub})^2}\right] \right\}.$$

Next, consider the no feedback case, where by Lemma S.5,

$$V^{NF} = \mathbb{E}_0 \int_0^T \left[-(\theta - a_t)^2 - (a_t - \hat{a}_t)^2 \right] dt = -\int_0^T (1-\alpha)^2 \gamma^o dt - \int_0^T \alpha^2 \gamma_t^{NF} dt. \quad \text{(S.17)}$$

Consider the first term on the RHS of (S.17), which reduces to $-T(1-\alpha)^2\gamma^o$. From Lemma S.2 we have $-T(1-\alpha)^2\gamma^o = -T\gamma^o \left(\frac{\rho}{1+\rho}\right)^2 = -\sigma_Y^2 \rho(1-\rho)$, where the last equality is simply a rearrangement of the equation $q(\gamma_T^{NF}) = 0$ using $\gamma_T^{NF} \equiv \rho\gamma^o$.

The second term on the RHS of (S.17) can be rewritten as

$$\int_0^T \sigma_Y^2 \frac{\dot{\gamma}_t^{NF}}{\gamma_t^{NF}} dt = \sigma_Y^2 (\ln \gamma_T - \ln \gamma_0) = \sigma_Y^2 \ln \frac{\gamma_T^{NF}}{\gamma^o} = \sigma_Y^2 \ln \rho,$$

and hence (S.17) is equivalent to $-\sigma_Y^2\{\rho(1-\rho) - \ln \rho\}$, as desired.

Next, we prove that $V^{NF} < V^{Pub}$ for all $T, \gamma^o, \sigma_Y^2 > 0$. Substituting in the expressions for γ_F^{Pub}, using $\tilde{T} := \frac{T\gamma^o}{\sigma_Y^2}$ and $\rho := \frac{\gamma_T^{NF}}{\gamma^o}$ and that the equation $q(\gamma_T^{NF}) = 0$ is equivalent to

$\tilde{T} = \frac{(1-\rho)(1+\rho)^2}{\rho}$, and simplifying, we obtain

$$V^{NF} - V^{Pub} = \sigma_Y^2 \Bigg\{ \rho(\rho-1) + \ln\rho + 1 + \tilde{T}/2 - \frac{\sqrt{4+\tilde{T}^2}}{2} \\ - \ln\left[8\left(-\tilde{T} + \sqrt{4+(\tilde{T})^2}\right)\right] + 2\ln\left[2 - \tilde{T} + \sqrt{4+(\tilde{T})^2}\right]\Bigg\} = \frac{\sigma_Y^2}{2\rho} f(\rho),$$

where $f(x) := A_1(x) + 2x\ln\left(\frac{A_2(x)^2}{A_3(x)}\right)$, for $A_1(x) := x^3 - 3x^2 + 3x + 1 - z(x), A_2(x) := x^3 + x^2 + x - 1 + z(x), A_3(x) := 8[x^3 + x^2 - x - 1 + z(x)]$ and $z(x) := \sqrt{4x^2 + (1-x)^2(1+x)^4}$.

We now show that $f(x) < 0$ for all $x \in (0,1)$, so that in particular $f(\rho) < 0$, from which the desired result follows. We begin by showing that $A_2(x) > 0$ and $A_3(x) > 0$ for all $x > 0$. By inspection, for all $x > 0$, we have $A_2(x) > A_3(x)/8$, and

$$A_3(x)/8 = (x-1)(x+1)^2 + \sqrt{4x^2 + (1-x)^2(1+x)^4} > (x+1)^2\left[x - 1 + \sqrt{(1-x)^2}\right] \geq 0.$$

Next, we apply the inequality $\ln(y) \leq 2\left(y^{\frac{1}{2}} - 1\right)$ for $y > 0$ using $y = \frac{A_2(x)^2}{A_3(x)} > 0$ to obtain

$$f(x) \leq A_1(x) + 4x\left(\frac{A_2(x)}{\sqrt{A_3(x)}} - 1\right). \tag{S.18}$$

For $x > 0$, the RHS of (S.18) is negative if and only if

$$A_2(x)/\sqrt{A_3(x)} < -A_1/(4x) + 1. \tag{S.19}$$

For $x \in (0,1)$, the RHS of (S.19) is strictly positive:

$$-\frac{A_1}{4x} + 1 = \frac{1}{4x}\left[\sqrt{4x^2 + (1-x)^2(1+x)^4} - x^3 + 3x^2 + x - 1\right] \\ > \frac{1}{4x}\left[(1-x)(1+x)^2 - x^3 + 3x^2 + x - 1\right] = \frac{1}{2}[1 + x(1-x)] > 0.$$

Hence, for $x \in (0,1)$, (S.19) is equivalent to

$$0 > A_2(x)^2 - A_3(x)\left(-\frac{A_1}{4x} + 1\right)^2 = \frac{2}{x^2}\left[(1-x)^2 A_4(x) + A_5(x)z(x)\right] \tag{S.20}$$

where $A_4(x) = x^6 - 4x^4 - x^3 + 4x^2 + 3x + 1$ and $A_5(x) = x^5 - 3x^4 + x^3 + 2x^2 - 1$. Now by Descartes' rule of signs, A_5 has 3 sign changes and at most 3 positive real roots, counting multiplicity. It is easy to verify that there is a double root at $x = 1$, that $A_5(2) = -1 < 0$, and that $\lim_{x\to+\infty} A_5(x) = +\infty$, so there is a positive root at some $x > 2$. This implies there are no roots in $(0,1)$. Since $A_5(0) = -1 < 0$, it follows that $A_5(x) < 0$ for all $x \in (0,1)$. Thus, without signing $A_4(x)$, it suffices to show that $(1-x)^2|A_4(x)| < -A_5(x)z(x)$, or equivalently (since $A_5(x) < 0$)

$$0 > (1-x)^4 A_4(x)^2 - z(x)^2 A_5(x)^2 = 4(1-x)^4 x^5 (x^4 - 2x^2 - 3x - 1). \tag{S.21}$$

Now for $x \in (0,1)$, we have in (S.21) $x^4 - 2x^2 - 3x - 1 < -2x^2 - 3x < 0$. Since the outside factor is positive, we have shown that $f(x) < 0$ for $x \in (0,1)$, and thus $V^{NF} < V^{Pub}$. □

S.2 Section 4: Omitted Proofs

Lemma S.6. *The process L is the belief about θ held by an outsider who observes only X. Moreover,* $\begin{pmatrix}\theta \\ \hat{M}_t\end{pmatrix} |\mathcal{F}_t^X \sim \mathcal{N}(M_t^{out}, \gamma_t^{out})$ *where* $M_t^{out} = \begin{pmatrix} L_t \\ L_t \end{pmatrix}$ *and* $\gamma_t^{out} = \begin{pmatrix} \frac{\gamma_{1t}}{1-\chi_t} & \frac{\gamma_{1t}\chi_t}{1-\chi_t} \\ \frac{\gamma_{1t}\chi_t}{1-\chi_t} & \frac{\gamma_{1t}\chi_t}{1-\chi_t} \end{pmatrix}$.

Proof. The outsider jointly filters the state $v_t = (\theta, \hat{M}_t)'$. For the evolution of the state and the signal, we adopt notation from Section 12.3 in Liptser and Shiryaev (1977). From the outsider's perspective, both players (and in particular player 2) are on the equilibrium path, and thus the outsider believes that v_t evolves as

$$dv_t = a_1(t, X^{out})v_t dt + b_1(t, X)dW_1(t) + b_2(t, X)dW_2(t),$$

where $a_1(t, X^{out}) := \begin{pmatrix} 0 & 0 \\ z_t & -z_t \end{pmatrix}$ for $z_t := \frac{(\nu\alpha_{3t})^2\gamma_{1t}}{\sigma_X^2} + \alpha_{3t}^2\gamma_{1t}$, $b_1(t, X) := \begin{pmatrix} 0 & 0 \\ 0 & \frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y} \end{pmatrix}$, $b_2(t, X) := \begin{pmatrix} 0 \\ \frac{\nu\alpha_{3t}\gamma_{1t}}{\sigma_X} \end{pmatrix}$, $W_1(t) := \begin{pmatrix} W_{11}(t) \\ Z_t^Y \end{pmatrix}$ and $W_2(t) := Z_t^X$, where $W_{11}(t)$ is a standard Brownian motion and $W_{11}(t), Z_t^Y$ and Z_t^X are mutually independent. The signal is

$$dX_t^{out} := dX_t - [\delta_{0t} + \delta_{2t}L_t + \nu(\alpha_{0t} + \alpha_{2t}L_t)]dt = A_1(t, X)v_t + B_1(t, X)W_1(t) + B_2(t, X)W_2(t),$$

where $A_1(t, X) := \begin{pmatrix} \nu\alpha_{3t} & \delta_{1t} \end{pmatrix}$, $B_1(t, X) := \begin{pmatrix} 0 & 0 \end{pmatrix}$ and $B_2(t, X) = \sigma_X$.

Hence, denoting $M_t^{out} = \begin{pmatrix} M_{t,1}^{out} \\ M_{t,2}^{out} \end{pmatrix}$ and $\gamma_t^{out} = \begin{pmatrix} \gamma_{t,11}^{out} & \gamma_{t,12}^{out} \\ \gamma_{t,21}^{out} & \gamma_{t,22}^{out} \end{pmatrix}$ and imposing $\gamma_{t,21}^{out} = \gamma_{t,12}^{out}$, we have from the standard filtering equations of Liptser and Shiryaev (1977, Theorem 12.7) that $\begin{pmatrix} \theta \\ \hat{M}_t \end{pmatrix} |\mathcal{F}_t^X \sim \mathcal{N}(M_t^{out}, \gamma_t^{out})$, where M^{out} and γ^{out} are the unique solutions to

$$\begin{aligned} dM_t^{out} &= a_1(t,X)M_t^{out} + \frac{1}{\sigma_X^2}\left[\begin{pmatrix} 0 \\ \nu\alpha_{3t}\gamma_{1t} \end{pmatrix} + \gamma_t^{out}\begin{pmatrix} \nu\alpha_{3t} \\ \delta_{1t} \end{pmatrix}\right] \times \cdots \\ &\qquad \cdots \times \left\{dX_t^{out} - (\nu\alpha_{3t}M_{t,1}^{out} + \delta_{1t}M_{t,2}^{out})dt\right\} \qquad \text{(S.22)} \\ \dot{\gamma}_t^{out} &= a_1(t,X)\gamma_t^{out} + \gamma_t^{out}a_1^* + \frac{1}{\sigma_X^2}\left[\begin{pmatrix} 0 \\ \nu\alpha_{3t}\gamma_{1t} \end{pmatrix} + \gamma_t^{out}\begin{pmatrix} \nu\alpha_{3t} \\ \delta_{1t} \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ \nu\alpha_{3t}\gamma_{1t} \end{pmatrix} + \gamma_t^{out}\begin{pmatrix} \nu\alpha_{3t} \\ \delta_{1t} \end{pmatrix}\right]^* \end{aligned}$$

with initial conditions $M_0^{out} = \begin{pmatrix} \mu \\ \mu \end{pmatrix}$ and $\gamma_0^{out} = \begin{pmatrix} \gamma^o & 0 \\ 0 & 0 \end{pmatrix}$.

Recall that γ_1 and χ satisfy $\dot{\gamma}_{1t} = -\frac{\alpha_{3t}^2}{\sigma_Y^2}\gamma_{1t}^2$ and $\dot{\chi}_t = \frac{\gamma_{1t}\alpha_{3t}^2(1-\chi_t)}{\sigma_Y^2} - \frac{\gamma_{1t}\delta_{1t}^2\chi_t^2}{\sigma_X^2}$ with initial conditions $\gamma_{10} = \gamma^o$ and $\chi_0 = 0$. It is straightforward to verify that $\gamma_t^{out} = \begin{pmatrix} \frac{\gamma_{1t}}{1-\chi_t} & \frac{\gamma_{1t}\chi_t}{1-\chi_t} \\ \frac{\gamma_{1t}\chi_t}{1-\chi_t} & \frac{\gamma_{1t}\chi_t}{1-\chi_t} \end{pmatrix}$ satisfies the γ^{out}-ODE above along with given initial condition. Moreover, γ_t^{out} is positive semidefinite as its leading principal minors are positive multiples of 1 and $\chi - \chi^2 > 0$.

Next, substitute given the solution γ_t^{out} into (S.22) and subtract the equation for the second component from its first to obtain the following SDE for $\bar{M} := M_1^{out} - M_2^{out}$: $d\bar{M}_t = -\Sigma\bar{M}_t\alpha_{3t}^2\gamma_{1t}$ with initial condition $\bar{M}_0 = 0$. Now if $\bar{M}_t > 0$, then $d\bar{M}_t < 0$, giving us a contradiction; likewise for the case $\bar{M}_t < 0$. It follows that $\bar{M}_t = 0$, and thus $M_{t,1}^{out} = M_{t,2}^{out}$, for all $t \geq 0$. Substituting this back into (S.22), we have

$$\begin{aligned} dM_{t,1}^{out} &= \frac{\gamma_{1t}(\nu\alpha_{3t} + \delta_{1t}\chi_t)}{\sigma_X^2(1-\chi_t)}(dX_t^{out} - (\nu\alpha_{3t} + \delta_{1t})M_{t,1}^{out}dt) \\ &= \frac{\gamma_{1t}(\nu\alpha_{3t} + \delta_{1t}\chi_t)}{\sigma_X^2(1-\chi_t)}[dX_t - (\nu\alpha_{0t} + \delta_{0t} + M_{1,t}^{out}(\nu\alpha_{3t} + \delta_{1t}) + L_t(\nu\alpha_{2t} + \delta_{2t}))dt] \qquad \text{(S.23)} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} dL_t &= \frac{L_t[\hat{\mu}_{1t} + \hat{\mu}_{2t} + \hat{\mu}_{3t}]dt + \hat{\mu}_{0t}dt + \hat{B}_t dX_t}{1-\chi_t} \\ &= \frac{\gamma_{1t}(\nu\alpha_{3t} + \delta_{1t}\chi_t)}{\sigma_X^2(1-\chi_t)}[dX_t - (\delta_{0t} + \nu\alpha_{0t} + L_t[\nu(\alpha_{2t} + \alpha_{3t}) + \delta_{1t} + \delta_{2t}])dt] \qquad \text{(S.24)} \end{aligned}$$

Hence $\bar{L}_t := M_{t,1}^{out} - L_t$ satisfies $d\bar{L}_t = -\frac{\gamma_{1t}(\nu\alpha_{3t}+\delta_{1t}\chi_t)}{\sigma_X^2(1-\chi_t)}\bar{L}_t(\nu\alpha_{3t} + \delta_{1t})$ with initial condition $\bar{L}_0 = \mu - \mu = 0$. We conclude that $\bar{L}_t = 0$, and thus $L_t = M_{t,1}^{out} = M_{t,2}^{out}$, for all $t \geq 0$. □

Proof of Lemma 4: We first derive a candidate mapping for the general case of a drift $\hat{a}_t + \nu a_t$, $\nu \in [0,1]$, in X. Suppose $\delta_1 = \hat{u}_{a\hat{a}}\alpha_3$. The χ-ODE for $\nu \in [0,1]$ boils down to

$$\dot{\chi}_t = \gamma_t \alpha_{3t}^2 \left(\left[\frac{\nu^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2} \right] (1 - \chi_t) - \frac{(\nu + \hat{u}_{a\hat{a}}\chi_t)^2}{\sigma_X^2} \right) =: -\gamma_t \alpha_{3t}^2 Q(\chi_t).$$

If $f : [0, \bar{\chi}) \to [0, \gamma^o]$, some $\bar{\chi} \in (0,1]$, is differentiable and $f(\chi_t) = \gamma_t$ for all $t \geq 0$, then $f'(\chi_t)\dot{\chi}_t = \dot{\gamma}_t$. When $\alpha_{3t} \neq 0$, $\frac{f'(\chi_t)}{f(\chi_t)} = \frac{\Sigma}{Q(\chi_t)}$. Hence, we solve the ODE $\frac{f'(\chi)}{f(\chi)} = \frac{\Sigma}{Q(\chi)}$ for $\chi \in (0, \bar{\chi})$ where $f(0) = \gamma^o$.

To this end, let $c_2 := \frac{\sqrt{b^2 + 4(\hat{u}_{a\hat{a}})^2/[\sigma_X \sigma_Y]^2} - b}{2(\hat{u}_{a\hat{a}}/\sigma_X)^2}$ and $-c_1 := \frac{-\sqrt{b^2 + 4(\hat{u}_{a\hat{a}})^2/[\sigma_X \sigma_Y]^2} - b}{2(\hat{u}_{a\hat{a}}/\sigma_X)^2}$, where $b := [\nu^2/\sigma_X^2 + 1/\sigma_Y^2] + 2\nu\hat{u}_{a\hat{a}}/\sigma_X^2$, be the roots of the convex quadratic Q above.

Clearly, $-c_1 < 0 < c_2$. Also, $c_2 \leq 1$ as $Q(1) \geq 0$; and when $\hat{u}_{a\hat{a}} \neq -\nu$ (which specializes to $\hat{u}_{a\hat{a}} \neq 0$, as in the lemma statement that uses $\nu = 0$), we have $c_2 < 1$. Thus, $\frac{\Sigma}{Q(\chi)} = -\frac{\sigma_X^2 \Sigma}{(\hat{u}_{a\hat{a}})^2(c_1 + c_2)} \left[\frac{1}{\chi + c_1} - \frac{1}{\chi - c_2} \right]$ is well defined (and negative) over $[0, c_2)$ with $1/(\chi + c_1) > 0$ and $-1/(\chi - c_2) > 0$ over the same domain. We can then set $\bar{\chi} = c_2$ and solve $\int_0^\chi \frac{f'(s)}{f(s)} ds = -\frac{\sigma_X^2 \Sigma}{(\hat{u}_{a\hat{a}})^2(c_1 + c_2)} \log\left(\frac{\chi + c_1}{c_2 - \chi} \frac{c_2}{c_1} \right)$, which yields the decreasing function $f(\chi) = f(0) \left(\frac{c_1}{c_2} \right)^{1/d} \left(\frac{c_2 - \chi}{\chi + c_1} \right)^{1/d}$, where $1/d = \sigma_X^2 \Sigma / [(\hat{u}_{a\hat{a}})^2 (c_1 + c_2)] > 0$. Imposing $f(0) = \gamma^o$ and inverting yields $\chi(\gamma) = f^{-1}(\gamma)$ as given in the lemma. Note that $\chi(\gamma^o) = 0$ and $\chi(0) = c_2$.

We now verify, for $\nu = 0$, that $\chi(\gamma)$ satisfies the χ-ODE (even when $\alpha_3 = 0$). We have

$$\frac{d(\chi(\gamma_t))}{dt} = \frac{\alpha_{3t}^2 \gamma_t}{\sigma_Y^2 [c_1 + c_2(\gamma/\gamma^o)^d]^2} c_1 c_2 d [c_1 + c_2] \left(\frac{\gamma_t}{\gamma^o} \right)^d.$$

By construction, moreover, $c_1 c_2 = c_1 - c_2 = \frac{\sigma_X^2}{\sigma_Y^2 (\hat{u}_{a\hat{a}})^2}$, which follows from equating the first- and zero-order coefficients in $Q(\chi) = \hat{u}_{a\hat{a}}^2 \chi^2/\sigma_X^2 + \chi/\sigma_Y^2 - 1/\sigma_Y^2 = \hat{u}_{a\hat{a}}^2 (\chi - c_2)(\chi + c_1)/\sigma_X^2$. Thus, $dc_1 c_2 = c_1 + c_2$. On the other hand,

$$\frac{[\hat{u}_{a\hat{a}}\chi(\gamma)]^2}{\sigma_X^2} = \frac{\hat{u}_{a\hat{a}}^2}{\sigma_X^2} \left[c_1 c_2 \frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2(\gamma/\gamma^o]^d} \right]^2 = \frac{c_1^2(1 - c_2)}{\sigma_Y^2} \left[\frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2(\gamma/\gamma^o)^d} \right]^2$$

where we used that $c_1^2 c_2^2/\sigma_X^2 = c_1^2(1 - c_2)/\sigma_Y^2$ follows from $\hat{u}_{a\hat{a}}^2 c_2^2/\sigma_X^2 = (1 - c_2)/\sigma_Y^2$ by definition of c_2. Thus, the right-hand side of the χ-ODE evaluated at our candidate $\chi(\gamma)$ satisfies

$$\gamma_1 \alpha_3^2 \left(\frac{1 - \chi}{\sigma_Y^2} - \frac{(\hat{u}_{a\hat{a}}\chi)^2}{\sigma_X^2} \right) \Bigg|_{\chi = \chi(\gamma)} = \frac{\alpha_3^2 \gamma_1}{\sigma_Y^2} \left(1 - \chi - c_1^2(1 - c_2) \left[\frac{1 - (\gamma/\gamma^o)^d}{c_1 + c_2(\gamma/\gamma^o)^d} \right]^2 \right).$$

Thus, using that $c_1 c_2 d = c_1 + c_2$ in our expression for $d(\chi(\gamma_t))/dt$, it suffices to show that

$$[c_1 + c_2]^2 \left(\frac{\gamma_t}{\gamma^o}\right)^d = (1-\chi)[c_1 + c_2(\gamma/\gamma^o)^d]^2 - c_1^2(1-c_2)[1-(\gamma/\gamma^o)^d]^2.$$

Using that $\chi[c_1 + c_2(\gamma/\gamma^o)^d] = 1 - (\gamma/\gamma^o)$, it is easy to conclude that this equality reduces to three equations $0 = c_1^2 - c_1^2 c_2 - c_1^2 + c_1^2 c_2$, $(c_1+c_2)^2 = 2c_1c_2 - c_1c_2(c_2 - c_1) + 2c_1^2(1-c_2)$ and $0 = c_2^2 + c_1 c_2^2 - c_1^2(1-c_2)$, capturing the conditions on the constant, $(\gamma/\gamma^o)^d$ and $(\gamma/\gamma^o)^{2d}$, respectively. The first condition is trivially satisfied. As for the third, by the definition of c_1 and c_2 we have that $c_2^2/(1-c_2) = \sigma_X^2/(\hat{u}_{a\hat{a}}\sigma_Y)^2 = c_1^2/(1+c_1)$. Thus, $c_1^1(1-c_2) = c_2^2(1+c_1)$, and the result follows. By canceling common terms, the second condition is also a rearrangement of this identity. Thus, $\chi(\gamma)$ as postulated satisfies the χ-ODE; by uniqueness, $\chi = \chi(\gamma)$.

Finally, when $\hat{u}_{a\hat{a}} = 0$, we have that $\delta_1 \equiv 0$, and the χ-ODE reduces to $\dot{\chi} = \alpha_{3t}^2\gamma_t(1-\chi_t)/\sigma_Y^2$, $\chi_0 = 0$. It is then easy to verify that $\chi(\gamma) = 1 - \gamma_t/\gamma^o$ satisfy the ODE, and we conclude again by uniqueness.

S.2.1 Proof of Theorem 1

It is convenient to change variables using $\tilde{v}_i := \gamma v_i$ for $i = 6, 8$:

$$\dot{\tilde{v}}_{6t} = \gamma_t \left\{ -\beta_{2t}^2 - 2\beta_{1t}\beta_{2t}(1-\chi_t) + \beta_{1t}^2(1-\chi_t)^2 + \tilde{v}_{6t}\alpha_t^2\left[\frac{1}{\sigma_Y^2} - \frac{2\chi_t}{\sigma_X^2(1-\chi_t)}\right]\right\} \tag{S.25}$$

$$\dot{\tilde{v}}_{8t} = \gamma_t \left\{ 2\beta_{2t} + 2(1-2\alpha_t)\beta_{1t}(1-\chi_t) + 4\beta_{1t}^2\chi_t(1-\chi_t) + \tilde{v}_{8t}\alpha_t^2\left[\frac{1}{\sigma_Y^2} - \frac{\chi_t}{\sigma_X^2(1-\chi_t)}\right]\right\} \tag{S.26}$$

$$\dot{\beta}_{1t} = \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1-\chi_t)}\left\{-2\sigma_X^2(\alpha_t - \beta_{1t})\beta_{1t}(1-\chi_t) + 2\sigma_Y^2\alpha_t\chi_t(\beta_{2t}[1+2\beta_{1t}\chi_t] - \beta_{1t}[1-\chi_t]) + \alpha_t^2\beta_{1t}\chi_t\tilde{v}_{8t}\right\} \tag{S.27}$$

$$\dot{\beta}_{2t} = \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1-\chi_t)}\left\{-2\sigma_X^2\beta_{1t}^2(1-\chi_t)^2 - 2\sigma_Y^2\alpha_t\beta_{2t}\chi_t^2(1-2\beta_{2t}) + \alpha_t^2\chi_t(2\tilde{v}_{6t} + \beta_{2t}\tilde{v}_{8t})\right\} \tag{S.28}$$

$$\dot{\beta}_{3t} = \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1-\chi_t)}\left\{2\sigma_X^2\beta_{1t}(1-\chi_t)\beta_{3t} - 2\sigma_Y^2\alpha_t\beta_{2t}\chi_t^2(1-2\beta_{3t}) + \alpha_t^2\beta_{3t}\chi_t\tilde{v}_{8t}\right\} \tag{S.29}$$

$$\dot{\gamma}_t = \frac{\gamma_t^2\alpha_t^2}{\sigma_Y^2} \tag{S.30}$$

with initial values $\tilde{v}_{60} = \tilde{v}_{80} = 0$, $\beta_{10} = \frac{1}{2(2-\chi_0)}$, $\beta_{20} = \frac{1-\chi_0}{2(2-\chi_0)}$, $\beta_{30} = \frac{1}{2}$ and $\gamma_0 = \gamma^F$, and where $\chi_t = \chi(\gamma_t)$, with $\chi(\cdot)$ defined in Lemma 4. From that lemma, this function takes values in $(-c_1, c_2]$, with $c_2 < 1$, when $\gamma \in [0, +\infty)$, and so the denominators in System 1 are well defined; we use $\bar{\chi} := c_2$ in what follows. Also, the right hand sides above are C^1 over $\{(\tilde{v}_6, \tilde{v}_8, \beta_1, \beta_2, \beta_3, \gamma) \in \mathbb{R}^5 \times \mathbb{R}_+\}$, and hence, if a solution exists over $[0, T]$, it is unique. We will show that there exists $\gamma^F \in (0, \gamma^o)$ such that a (unique) solution to System 1 exists and it satisfies $\gamma_T(\gamma^F) = \gamma^o$, with the properties stated in the theorem. For such γ^F, since γ is nondecreasing going back in time, we have $\gamma > 0$ and can recover $v_6 = \tilde{v}_6/\gamma$ and $v_8 = \tilde{v}_8/\gamma$.

It is easy to verify that $\alpha = \beta_3 + \beta_1\chi$ satisfies $\dot{\alpha}_t = \frac{\alpha_t^3\gamma_t\chi_t}{2\sigma_X^2\sigma_Y^2(1-\chi_t)}\{4\sigma_Y^2\beta_2\chi_t + \alpha_t\tilde{v}_{8t}\}$ with initial condition $\alpha_0 = \frac{1}{2-\chi(\gamma^F)} > 0$. By our comparison theorem applied to α and 0, we have $\alpha > 0$. The functions $\tilde{v}_6^{cand} := \frac{\sigma_Y^2[-1+2\beta_1(1-\chi)+\alpha]}{\alpha} - \frac{\tilde{v}_8}{2}$ and $\beta_2^{cand} := 1 - \beta_1 - \beta_3$ are then well defined, and it can be checked that if $(\tilde{v}_6, \tilde{v}_8, \beta_1, \beta_2, \beta_3, \gamma)$ is a solution to System 1, then $(\tilde{v}_6^{cand}, \tilde{v}_8, \beta_1, \beta_2^{cand}, \beta_3, \gamma)$ is also a solution. Hence, by uniqueness, $\tilde{v}_6 = \tilde{v}_6^{cand}$ and $\beta_2 = \beta_2^{cand}$, and so doing the corresponding replacements in the $(\tilde{v}_8, \beta_1, \beta_3, \gamma)$ ODEs we obtain the reduced IVP ("System 2")

$$\begin{aligned}\dot{\tilde{v}}_{8t} &= \gamma_t\left\{2\beta_{2t}^{cand} + 2(1-2\alpha_t)\beta_{1t}(1-\chi_t) + 4\beta_{1t}^2\chi_t(1-\chi_t)\right.\\ &\qquad \left.+\tilde{v}_{8t}\alpha_t^2\left[1/\sigma_Y^2 - \chi_t/(\sigma_X^2[1-\chi_t])\right]\right\}\end{aligned} \tag{S.31}$$

$$\begin{aligned}\dot{\beta}_{1t} &= \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1-\chi_t)}\left\{-2\sigma_X^2(\alpha_t - \beta_{1t})\beta_{1t}(1-\chi_t)\right.\\ &\qquad \left.+2\sigma_Y^2\alpha_t\chi_t(\beta_{2t}^{cand}[1+2\beta_{1t}\chi_t] - \beta_{1t}[1-\chi_t]) + \alpha_t^2\beta_{1t}\chi_t\tilde{v}_{8t}\right\}\end{aligned} \tag{S.32}$$

$$\begin{aligned}\dot{\beta}_{3t} &= \frac{\alpha_t\gamma_t}{2\sigma_X^2\sigma_Y^2(1-\chi_t)}\left\{2\sigma_X^2\beta_{1t}(1-\chi_t)\beta_{3t} - 2\sigma_Y^2\alpha_t\beta_{2t}^{cand}\chi_t^2(1-2\beta_{3t})\right.\\ &\qquad \left.+\alpha_t^2\beta_{3t}\chi_t\tilde{v}_{8t}\right\},\end{aligned} \tag{S.33}$$

along with (S.30), and initial conditions $\tilde{v}_{80} = 0$, $\beta_{10} = \frac{1}{2(2-\chi(\gamma^F))}$, $\beta_{30} = \frac{1}{2}$ and $\gamma_0 = \gamma^F$.

Note that if $\gamma^F = 0$, System 2 has the (unique) solution $\chi = \bar{\chi}$, $\beta_1 = \frac{1}{2(2-\bar{\chi})}$, $\beta_3 = \frac{1}{2}$ (so that $\alpha = \frac{1}{2-\bar{\chi}}$) and $\gamma = 0$, with $\tilde{v}_8$ obtained from integration of its ODE. (This does not correspond to a solution to the BVP, since $\gamma_T(0) = 0 < \gamma^o$.) Next, define

$$\bar{\gamma} := \sup\{\tilde{\gamma}^F > 0 \mid \text{a solution to System 2 exists over } [0, T] \text{ for all } \gamma^F \in (0, \tilde{\gamma}^F)\}.$$

Since the right-hand side of the equations that comprise System 2 are of class C^1, the solution is unique when it exists, and there is continuous dependence of the solution on the initial conditions; in particular, the terminal value γ_T is continuous in γ^F (see Theorem on page 397 in Hirsch et al. (2004)). Hence if there exists $\gamma^F \in (0, \bar{\gamma})$ such that $\gamma_T(\gamma^F) \geq \gamma^o$, by the intermediate value theorem there exists a $\gamma^F \in (0, \bar{\gamma})$ such that $\gamma_T(\gamma^F) = \gamma^o$, allowing us to construct a solution to System 1.

Suppose then that for all $\gamma^F \in (0, \bar{\gamma})$, $\gamma_T(\gamma^F) < \gamma^o$. In particular, because γ_t is nondecreasing in the backward system for any initial condition, we have that $\gamma_t \in (0, \gamma^o)$ and by Lemma 4, $\chi_t \in (0, \bar{\chi})$ for all $t \in [0, T]$ when $\gamma^F \in (0, \bar{\gamma})$. To reach a contradiction, it suffices to show that the solution to System 2 can be bounded uniformly over $\gamma^F \in (0, \bar{\gamma})$ for times T as in the theorem. Indeed, given uniform bounds, we first claim that a solution to System 2 for $\gamma^F = \bar{\gamma}$ must exist over $[0, T]$. To see this, let $[0, \tilde{T})$ denote the maximal interval of existence, and suppose by way of contradiction that $\tilde{T} \in (0, T]$. Thus, there must be some function $x(\cdot, \bar{\gamma})$ which explodes at $\tilde{T}$, and so, for $\tilde{t} \in (0, \tilde{T})$ sufficiently close to $\tilde{T}$, we have $x(\tilde{t}, \bar{\gamma}) \notin [-\underline{K}^x, \overline{K}^x]$, with $[-\underline{K}^x, \overline{K}^x]$ the interval that uniformly bounds $x(t, \gamma^F)$ over $(t, \gamma^F) \in [0, T] \times (0, \bar{\gamma})$. But for any sequence $(\gamma_n^F)_{n \in \mathbb{N}}$ taking values in $(0, \bar{\gamma})$ such that $\gamma_n^F \uparrow \bar{\gamma}$, by continuity of solutions with respect to initial conditions, we have $x(\tilde{t}, \bar{\gamma}) = \lim_{n \to \infty} x(\tilde{t}, \gamma_n^F) \in [-\underline{K}^x, \overline{K}^x]$, a contradiction. We conclude that a solution to System 2 for $\gamma^F = \bar{\gamma}$ must exist over $[0, T]$, and hence, by the extensibility of the solutions (Theorem on page 397 in Hirsch et al. (2004)), that a solution must also exist for all $\gamma^F \in [\bar{\gamma}, \bar{\gamma} + \epsilon)$, some $\epsilon > 0$, thereby violating the definition of $\bar{\gamma}$ as a supremum.

To obtain cleaner expression for the uniform bounds, let us analyze a "centered" system that arises from decomposing β_1 and β_3 as sums of forward-looking and myopic components, and show that both of these components are uniformly bounded. Specifically, define $\beta_{1t}^m := \frac{1}{2(2-\chi_t)}$, $\beta_{3t}^m = \frac{1}{2}$ and $\beta_{it}^f := \beta_{it} - \beta_{it}^m$ for all $t \in [0, T]$, $i = 1, 3$; since $\tilde{v}_8$ starts (going backwards) at zero, it coincides with its own forward-looking component by definition. Observe that for $\gamma^F \in (0, \bar{\gamma})$, β_{1t}^m is uniformly bounded by $[1/4, 1/2] \subset [0, 1]$, as $\chi_t = \chi(\gamma_t) \in [0, 1]$ for all $t \in [0, T]$, and trivially $\beta_{3t}^m \in [0, 1]$. Thus, it suffices to show that the solutions $(\tilde{v}_8, \beta_1^f, \beta_3^f)$ are uniformly bounded by some $[-K, K]^3$. (Recall that γ is already bounded by $[0, \gamma^o]$.) Hence, we define one final IVP ("System 3") in $(\tilde{v}_8, \beta_1^f, \beta_3^f, \gamma)$ to be (S.30), (S.31), and

$$
\begin{aligned}
\dot{\beta}_{1t}^f = & -\frac{\dot{\chi}_t}{2(2-\chi_t)^2} + \frac{\alpha_t \gamma_t}{2\sigma_X^2 \sigma_Y^2 (1-\chi_t)} \Big\{ -2\sigma_X^2 (\alpha_t - [\beta_{1t}^f + \beta_{1t}^m])[\beta_{1t}^f + \beta_{1t}^m](1-\chi_t) \\
& + 2\sigma_Y^2 \alpha_t \chi_t \Big([1 - (\beta_{1t}^f + \beta_{1t}^m) - (\beta_{3t}^f + \beta_{3t}^m)][1 + 2(\beta_{1t}^f + \beta_{1t}^m)\chi_t] \Big) \\
& - 2\sigma_Y^2 \alpha_t \chi_t (\beta_{1t}^f + \beta_{1t}^m)(1-\chi_t) + \alpha_t^2 (\beta_{1t}^f + \beta_{1t}^m)\chi_t \tilde{v}_{8t} \Big\}, \\
=: & \, h^{\beta_1^f}(\beta_{1t}^f, \beta_{1t}^m, \beta_{3t}^f, \beta_{3t}^m, \tilde{v}_{8t}, \gamma_t)
\end{aligned}
\tag{S.34}
$$

$$
\begin{aligned}
\dot{\beta}_{3t}^f = & \frac{\alpha_t \gamma_t}{2\sigma_X^2 \sigma_Y^2 (1-\chi_t)} \Big\{ 2\sigma_X^2 (\beta_{1t}^f + \beta_{1t}^m)(1-\chi_t)(\beta_{3t}^f + \beta_{3t}^m) + \alpha_t^2 (\beta_{3t}^f + \beta_{3t}^m)\chi_t \tilde{v}_{8t} \\
& + 4\sigma_Y^2 \alpha_t \beta_{3t}^f \chi_t^2 [1 - (\beta_{1t}^f + \beta_{1t}^m) - (\beta_{3t}^f + \beta_{3t}^m)] \Big\} \\
=: & \, h^{\beta_3^f}(\beta_{1t}^f, \beta_{1t}^m, \beta_{3t}^f, \beta_{3t}^m, \tilde{v}_{8t}, \gamma_t)
\end{aligned}
\tag{S.35}
$$

subject to initial conditions $\tilde{v}_{80} = 0$, $\beta_{10}^f = 0$, $\beta_{30}^f = 0$ and $\gamma_0 = \gamma^F \in (0, \bar{\gamma})$, where $\alpha_t = [\beta_{3t}^f + \beta_{3t}^m] + [\beta_{1t}^f + \beta_{1t}^m]\chi_t$. Define $h^{\tilde{v}_8}(\beta_{1t}^f, \beta_{1t}^m, \beta_{3t}^f, \beta_{3t}^m, \tilde{v}_{8t}, \gamma_t)$ as the RHS of (S.31) with $\beta_{it}^f + \beta_{it}^m$

substituted for β_{it}, $i = 1, 3$.

Define $\bar{\alpha} = (K+1)\bar{\chi} + (K+1)$, where we suppress dependence on K. Next, for $x \in \{\tilde{v}_8, \beta_1^f, \beta_3^f\}$ define $\bar{h}^x : \mathbb{R}^2_{++} \to R_{++}$ as follows:

$$\begin{aligned}
\bar{h}^{\tilde{v}_8}(\gamma^o; K) &:= \gamma^o \{2[1+2(K+1)] + 2(1+2\bar{\alpha})(K+1) \\
&\qquad +4(K+1)^2\bar{\chi} + K\bar{\alpha}^2 \left[1/\sigma_Y^2 + \bar{\chi}/(\sigma_X^2[1-\bar{\chi}])\right]\}
\end{aligned} \tag{S.36}$$

$$\begin{aligned}
\bar{h}^{\beta_1^f}(\gamma^o; K) &:= \frac{\bar{\alpha}^2\gamma^o \left[1/\sigma_Y^2 + \bar{\chi}/(\sigma_X^2[1-\bar{\chi}])\right]}{2(2-\bar{\chi})^2} + \frac{\bar{\alpha}\gamma^o}{2\sigma_X^2\sigma_Y^2(1-\bar{\chi})} \left\{2\sigma_X^2[\bar{\alpha}+K+1](K+1)\right. \\
&\qquad \left. +2\sigma_Y^2\bar{\alpha}\bar{\chi}([1+2(K+1)][1+2(K+1)\bar{\chi}] + K+1) + \bar{\alpha}^2K(K+1)\bar{\chi}\right\}
\end{aligned} \tag{S.37}$$

$$\begin{aligned}
\bar{h}^{\beta_3^f}(\gamma^o; K) &:= \frac{\bar{\alpha}\gamma^o}{2\sigma_X^2\sigma_Y^2(1-\bar{\chi})} \left\{2\sigma_X^2(K+1)^2 + \bar{\alpha}^2\bar{\chi}K(K+1)\right. \\
&\qquad \left. +4\sigma_Y^2\bar{\alpha}\bar{\chi}^2K[1+2(K+1)]\right\},
\end{aligned} \tag{S.38}$$

Define

$$T(\gamma^o) := \max_{K' \geq 0} \min_{x \in \{\tilde{v}_8, \beta_1^f, \beta_3^f\}} \frac{K'}{\bar{h}^x(\gamma^o; K')} \in (0, \infty),$$

and let $K \in (0, \infty)$ denote the arg max.[1] It is clear that given $T < T(\gamma^o)$, $(\tilde{v}_8, \beta_1^f, \beta_3^f)$ are uniformly bounded by $[-K, K]^3$. Indeed, suppose otherwise, and define $\tau = \inf\{t > 0 : (\tilde{v}_{8t}, \beta_{1t}^f, \beta_{3t}^f) \notin [-K, K]^3\}$; by supposition and continuity of the solutions, $\tau \in (0, T)$ and $|x_\tau| = K$, some $x \in \{\tilde{v}_8, \beta_1^f, \beta_3^f\}$. Now, by construction of $\bar{h}^x(\gamma^o; K)$, for all $t \in [0, \tau]$ and for each $x \in \{\tilde{v}_8, \beta_1^f, \beta_3^f\}$ we have $|\dot{x}_t| = |h^x(\beta_{1t}^f, \beta_{1t}^m, \beta_{3t}^f, \beta_{3t}^m, \tilde{v}_{8t}, \gamma_t)| < \bar{h}^x(\gamma^o; K)$, and thus by the triangle inequality, $|x_\tau| < 0 + \tau \cdot \bar{h}^x(\gamma^o; K) < T(\gamma^o)\bar{h}^x(\gamma^o; K) \leq K$, a contradiction.

We conclude that the solutions $(\tilde{v}_8, \beta_1^f, \beta_3^f, \gamma)$ to System 3 are uniformly bounded by $[-K, K]^3 \times [0, \gamma^o]$, and by another application of the triangle inequality the solutions $(\tilde{v}_8, \beta_1, \beta_3, \gamma)$ to System 2 are uniformly bounded by $[-K, K] \times [-K, K+1]^2 \times [0, \gamma^o]$. This gives us the desired contradiction of the definition of $\bar{\gamma}$ from before, so we conclude that for $T < T(\gamma^o)$, there exists $\gamma^F \in (0, \bar{\gamma})$ such that the solution to System 2 satisfies $\gamma_T(\gamma^F) = \gamma^o$. (Note that any such γ^F lies in $(0, \gamma^o)$, as γ is nondecreasing in the backward system.) In turn, there exists a solution to System 1 by setting $\tilde{v}_6 = \tilde{v}_6^{cand}$ and $\beta_2 = \beta_2^{cand}$.

Assume now that $T < T(\gamma^o)$ and consider any such γ^F as above and its induced solution to System 1 with $\gamma_T(\gamma^F) = \gamma^o$. To conclude the proof, we must characterize β_0 and the remaining value function coefficients. Since this step is the special case $\hat{u}_{\hat{a}a} = 1$, $\hat{u}_{\hat{a}\theta} = 0$ of an analogous step in the proof of Theorem 2, we refer the reader to that proof.

[1] Corner solutions are ruled out since the bounds $\bar{h}^x$ are strictly positive and grow faster than linearly in their second argument.

S.2.2 Remaining ODEs from Proof of Theorem 2

The arguments in the conclusion of the proof refer to the following ODEs:

$$\begin{aligned}
\dot{\beta}_{0t} = &-\frac{\gamma_t \chi_t}{2\sigma_X^2 \sigma_Y^2 (1-\chi_t)(1+\hat{u}_{\hat{a}\theta}\chi_t)} \Big\{ 4\hat{u}_{\hat{a}\theta}^2 \sigma_Y^2 \beta_{0t} \tilde{\beta}_{2t}(1-\chi_t)\chi_t + 2\alpha_t^2 \left[\hat{u}_{\hat{a}a}\hat{u}_{\hat{a}\theta} v_{3t}\gamma_t (1+\hat{u}_{\hat{a}\theta}\chi_t)\right] \\
&+ \hat{u}_{\hat{a}a}\alpha_t^3 \left[(1-\chi_t)\beta_{0t}(2[1-\hat{u}_{\hat{a}a}]\sigma_X^2 + \hat{u}_{\hat{a}a}\tilde{v}_{8t}) + \hat{u}_{\hat{a}a} v_{3t}\gamma_t (1+\hat{u}_{\hat{a}\theta}\chi_t)\right] \\
&+ \hat{u}_{\hat{a}\theta}\alpha_t \left[\hat{u}_{\hat{a}\theta} v_{3t}\gamma_t(1+\hat{u}_{\hat{a}\theta}\chi_t) + \beta_{0t}(1-\chi_t)(-2\hat{u}_{\hat{a}\theta}\sigma_X^2 + \hat{u}_{\hat{a}\theta}\tilde{v}_{8t} + 8\hat{u}_{\hat{a}a}\sigma_Y^2 \tilde{\beta}_{2t}\chi_t)\right] \\
&+ 2\alpha_t^2 \beta_{0t}(1-\chi_t)\left([1-2\hat{u}_{\hat{a}a}]\hat{u}_{\hat{a}\theta}\sigma_X^2 + \hat{u}_{\hat{a}a}[\hat{u}_{\hat{a}\theta}\tilde{v}_{8t} + 2\hat{u}_{\hat{a}a}\sigma_Y^2 \tilde{\beta}_{2t}\chi_t]\right)\Big\},\ \beta_{0T} = 0, \\
\dot{v}_{0t} = &\ \beta_{0t}^2 + (\hat{u}_{\hat{a}\theta} + \hat{u}_{\hat{a}a}\alpha_t)^2 \gamma_t \chi_t \left\{1 + \frac{\chi_t}{\sigma_X^2}\left[-\tilde{v}_{6t} + \sigma_Y^2(\hat{u}_{\hat{a}\theta} + \hat{u}_{\hat{a}a}\alpha_t - 2\tilde{\beta}_{2t})/\alpha_t\right]\right\},\ v_{0T} = 0, \\
\dot{v}_{1t} = &-2\beta_{0t},\ v_{1T} = 0, \\
\dot{v}_{3t} = &\ 2\beta_{0t}(\beta_{1t} + \tilde{\beta}_{2t})(1-\chi_t) + \frac{v_{3t}(\hat{u}_{\hat{a}\theta} + \hat{u}_{\hat{a}a}\alpha_t)^2 \gamma_t \chi_t}{\sigma_X^2(1-\chi_t)},\ v_{3T} = 0, \quad \text{and} \\
\dot{v}_{4t} = &\ 1 - 2\beta_{3t}^2,\ v_{4T} = 0.
\end{aligned}$$

S.3 Section 5: Omitted Proofs

S.3.1 Proof of Proposition 5, Part (i)

We analyze the public and no-feedback cases, and then we compare learning and payoffs.

Public Case

We look for an equilibrium of the form $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{3t}\theta$, where $M_t = \hat{M}_t$ is publicly known, with value function $V(\theta, m, t) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\theta^2 + v_{4t}m^2 + v_{5t}\theta m$.

The core (backward) system of ODEs is

$$(\dot{\beta}_{0t}, \dot{\beta}_{1t}, \dot{\beta}_{3t}, \dot{\gamma}_t) = (0, -\beta_{1t}\beta_{3t}^2\gamma_t^2/\sigma_Y^2, \beta_{1t}\beta_{3t}^2\gamma_t^2/\sigma_Y^2, \beta_{3t}^2\gamma_t^2/\sigma_Y^2).$$

with initial conditions $\beta_{00} = 0, \beta_{10} = -\frac{\psi\gamma_0}{\sigma_Y^2} \leq 0, \beta_{30} = 1$ and $\gamma_0 = \gamma^F \in (0, \gamma^o)$.

Define $\tilde{\psi} := \psi\gamma^o/\sigma_Y^2$ and $\tilde{T} := T\gamma^o/\sigma_Y^2$. It is easy to verify that the system has a solution for each root $\rho^{Pub} \in (0,1)$ of the cubic $g^{Pub}(\rho) := -\tilde{T}\tilde{\psi}\rho^2(1-\rho) + \rho(1+\tilde{T}) - 1 = 0$, where $(\beta_{0t}, \beta_{1t}, \beta_{3t}) = (0, \beta_{10}\gamma^F/\gamma_t, 1 + \beta_{10}(1-\gamma^F/\gamma_t)$ where $\gamma_t = \frac{\gamma^F[\sigma_Y^4 + t\psi(\gamma^F)^2]}{\sigma_Y^4 - t\gamma^F(-\gamma^F\psi + \sigma_Y^2)}$ and $\gamma^F = \rho^{Pub}\gamma^o$. Such a root always exists since $g^{Pub}(0) < 0 < g^{Pub}(1)$, and hence an LME exists. Now the quadratic $g^{Pub\prime}(\rho)$ has roots only if $\tilde{\psi} \geq 3(1+\tilde{T})/\tilde{T} > 3$. Hence, when $\psi < \sigma_Y^2/\gamma^o$ (i.e., $\tilde{\psi} < 1$), g^{Pub} is strictly increasing, and the root, and hence the LME, is unique.

No Feedback Case

We look for an equilibrium with $a_t = \beta_0\mu + \beta_1 M_t + \beta_3\theta$, where $M_t = \mathbb{E}_t^1[\hat{M}_t]$, with value function $V(t,\theta,m) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\theta^2 + v_{4t}m^2 + v_{5t}\theta ml$. The backward system is

$$(\dot{\beta}_{0t}, \dot{\beta}_{1t}, \dot{\beta}_{3t}, \dot{\gamma}_t) = \left(0, -\alpha_t\beta_{1t}[\beta_{3t}\gamma_t - \beta_{1t}\gamma_t(1-\chi_t)]/\sigma_Y^2, \alpha_t\beta_{1t}\beta_{3t}\gamma_t/\sigma_Y^2, \alpha_t^2\gamma_t^2/\sigma_Y^2\right)$$

with initial conditions $\beta_{00} = 0$, $\beta_{10} = -\frac{\psi\gamma_0}{\sigma_Y^2 + \psi\gamma_0\chi(\gamma_0)}$, $\beta_{30} = 1$, and $\gamma_0 = \gamma^F$, where $\chi(\gamma) := 1 - \gamma/\gamma^o$. Note that $\dot{\alpha}_t = 0$, so $\alpha = \alpha_0 = \frac{\sigma_Y^2}{\sigma_Y^2 + \psi\gamma_0\chi(\gamma_0)}$, and it suffices to solve the γ-ODE.

By standard bounding arguments, the system has a solution. With algebra, one can show that there is one solution for each root $\rho^{NF} \in (0,1)$ of the quintic $g^{NF}(\rho) := \tilde{T}\rho - (1-\rho)[1 + \tilde{\psi}\rho(1-\rho)]^2$, with $\gamma_t = \frac{\gamma^F\sigma_Y^2}{\sigma_Y^2 - \gamma^F\bar{\alpha}^2 t}$, where $\gamma^F = \rho^{NF}\gamma^o$, and with $\alpha = \bar{\alpha} = \alpha_0$ being a constant as above. Such a root always exists since $g^{NF}(0) < 0 < g^{NF}(1)$, and hence an LME exists. When $\psi < \sigma_Y^2/\gamma^o$, it can be verified that g^{NF} is strictly increasing when it crosses zero in $(0,1)$; hence its root, and the LME, is unique.

Learning and Payoff Comparisons

Lemma S.7. *If $\tilde{\psi} \in (0,1]$, then there is more learning in the public case for all $T > 0$.*

Proof. Let $\rho^x = \gamma_T^x/\gamma^o \in (0,1)$, where γ_T^x is the terminal value of γ in the BVP of case $x \in \{\text{public, no feedback}\}$. When $\tilde{\psi} \in (0,1]$, these values are the unique roots of

$$0 = g^{NF}(\rho) := \rho\tilde{T} - (1-\rho)[1 + \tilde{\psi}\rho(1-\rho)]^2 = \rho(1+\tilde{T}) - 1 - \tilde{\psi}\rho(1-\rho)^2[2 + \tilde{\psi}\rho(1-\rho)]$$
$$0 = g^{Pub}(\rho) := \rho(1+\tilde{T}) - 1 - \tilde{\psi}\tilde{T}\rho^2(1-\rho),$$

respectively. In particular, observe that $\rho^x > 1/(1+\tilde{T})$, $x \in \{\text{public, no feedback}\}$. Our goal is to show $\rho^{Pub} < \rho^{NF}$.

Now, using that $\rho^{Pub}(1+\tilde{T}) - 1 = \tilde{\psi}\tilde{T}(\rho^{Pub})^2(1-\rho^{Pub})$, we get that

$$g^{NF}(\rho^{Pub}) = \frac{\tilde{\psi}(1-\rho^{Pub})}{\tilde{T}}\left\{\tilde{T}^2(\rho^{Pub})^2 - (1-\rho^{Pub})[2\rho^{Pub}\tilde{T} + \rho^{Pub}(1+\tilde{T}) - 1]\right\}.$$

where $\frac{\tilde{\psi}(1-\rho^{Pub})}{\tilde{T}} > 0$. Thus, letting

$$Q(\rho) := \tilde{T}^2\rho^2 - (1-\rho)[2\rho\tilde{T} + \rho(1+\tilde{T}) - 1] = \rho^2(\tilde{T}^2 + 3\tilde{T} + 1) - \rho(3\tilde{T} + 2) + 1,$$

it suffices to show that $Q(\rho^{Pub}) < 0$, as $g^{NF}(\rho) < 0$ if and only if $\rho < \rho^{NF}$.

Observe that the roots of Q are given by $\rho_- := \frac{(3-\sqrt{5})\tilde{T}+2}{2(\tilde{T}^2+3\tilde{T}+1)}$ and $\rho_+ := \frac{(3+\sqrt{5})\tilde{T}+2}{2(\tilde{T}^2+3\tilde{T}+1)}$, and that $\rho_- < \frac{1}{1+\tilde{T}} < \rho_+$. Consequently, it suffices to show that $g^{Pub}(\rho_+) > 0$: this ensures that $\rho^{Pub} < \rho_+$, and since $\rho^{Pub} > \frac{1}{1+\tilde{T}} > \rho_-$, this implies that $Q(\rho^{Pub}) < 0$.

Straightforward algebraic manipulation yields that $g^{Pub}(\rho_+) > 0$ if and only if

$$\begin{aligned}\tilde{g}(\tilde{T},\tilde{\psi}) \;&:= 4(1+\tilde{T})[(3+\sqrt{5})\tilde{T}+2][\tilde{T}^2+3\tilde{T}+1]^2 - 8[\tilde{T}^2+3\tilde{T}+1]^3\\ &\qquad -\tilde{\psi}\tilde{T}^2[(3+\sqrt{5})\tilde{T}+2]^2[2\tilde{T}+(3-\sqrt{5})] > 0.\end{aligned}$$

The constraint is tightest when $\tilde{\psi}=1$, and $\tilde{g}(\tilde{T},1)$ can be written as $\tilde{T}\sum_{i=0}^{5} a_i\tilde{T}^i$ where all the $a_i > 0$. Hence, $\tilde{g}(\tilde{T},\tilde{\psi}) > 0$ whenever $\tilde{T} > 0$ and $\tilde{\psi} \in (0,1]$, concluding the proof. □

Let V^x denote the ex ante payoff to player 1 in the case $x \in \{Pub, NF\}$. First,

$$\begin{aligned}V^{Pub} &= \mathbb{E}_0\left[-\int_0^T (a_t-\theta)^2 dt - \psi M_T^2\right]\\ &= -\int_0^T \mathbb{E}_0\left[(\beta_{1t}M_t + [\beta_{3t}-1]\theta)^2\right] dt - \psi(\mu^2+\gamma^o-\gamma_T)\\ &= -\int_0^T \left[(\beta_{3t}-1)^2\gamma^o + \beta_{1t}^2(\gamma^o-\gamma_t) + 2\beta_{1t}(\beta_{3t}-1)(\gamma^o-\gamma_t)\right] dt - \tilde{\psi}\sigma_Y^2(1-\rho^{Pub}).\end{aligned}$$

Using the solutions for the coefficients and γ_t in terms of γ^F and carrying out the simplifications, we obtain $V^{Pub} = V^{Pub}(\rho^{Pub})$, where

$$V^{Pub}(\rho) := \sigma_Y^2\left\{-\tilde{\psi}(1-\rho) + \tilde{T}\tilde{\psi}\rho^2[-\tilde{\psi}(1-\rho)+1] + \ln\left(\frac{1-\rho}{\tilde{T}\rho}\right)\right\}.$$

In the no feedback case, note that $\mathbb{E}_0[M_t^2] = \mathbb{E}_0[(\chi_t\theta + (1-\chi_t)\mu)^2] = \mathbb{E}_0[\chi_t^2\theta^2] = \chi_t^2\gamma^o$. Hence, $\mathbb{E}_0[\hat{M}_t^2] = \mathbb{E}_0[(\hat{M}_t - M_t)^2] + \mathbb{E}_0[M_t^2] = \gamma_{2t} + \chi_t^2\gamma^o = \chi_t\gamma_t + \chi_t^2\gamma^o$.

Using $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{3t}\theta = \bar{a}\theta$, we now calculate

$$V^{NF} = \mathbb{E}_0\left[-\int_0^T (a_t-\theta)^2 dt - \psi(\chi_T\gamma_T + \chi_T^2\gamma^o)\right] = -(1-\bar{\alpha})^2\gamma^o T - \psi\chi_T(\gamma_T + \chi_T\gamma^o).$$

Expressing $\chi_T = 1-\gamma_T/\gamma^o$, γ_T and $\bar{\alpha}$ in terms of $\gamma^F = \gamma_T$, we have $V^{NF} = V^{NF}(\rho^{NF})$, where $V^{NF}(\rho) := \sigma_Y^2\left\{-\tilde{\psi}^2\rho(1-\rho)^3 - \tilde{\psi}(1-\rho)\right\}$.

Lemma S.8. *For $\tilde{\psi} \in (0,1]$, the long-run player is better off in the no feedback case than in the public case for all $T > 0$.*

Proof. We show that for such $\tilde{\psi}$, (i) $V^{Pub}(\rho^{Pub}) < V^{NF}(\rho^{Pub})$ and (ii) $V^{NF}(\rho)$ is increasing for $\rho \geq \rho^{Pub}$, so that $V^{Pub}(\rho^{Pub}) < V^{NF}(\rho^{NF})$.

Toward establishing (i), define $\tilde{V}(\rho) := V^{Pub}(\rho) - V^{NF}(\rho)$; we have

$$\tilde{V}(\rho) = \sigma_Y^2 \left\{ \tilde{T}\tilde{\psi}\rho^2[-\tilde{\psi}(1-\rho)+1] + \ln\left(\frac{1-\rho}{\tilde{T}\rho}\right) + \tilde{\psi}^2\rho(1-\rho)^3 \right\},$$

and our first goal is to show $\tilde{V}(\rho^{Pub}) < 0$. Since $\ln(x) < x - 1$ for $x > 0$, we have

$$\tilde{V}(\rho) < \sigma_Y^2 \left\{ \tilde{T}\tilde{\psi}\rho^2[-\tilde{\psi}(1-\rho)+1] + \left[\frac{1-\rho}{\tilde{T}\rho} - 1\right] + \tilde{\psi}^2\rho(1-\rho)^3 \right\} = \frac{\sigma_Y^2}{\tilde{T}\rho}\tilde{V}_2(\rho),$$

where $\tilde{V}_2(\rho) := \tilde{T}^2\tilde{\psi}\rho^3[1-\tilde{\psi}(1-\rho)] + 1 - \rho(1+\tilde{T}) + \tilde{T}\tilde{\psi}^2\rho^2(1-\rho)^3$, and so it suffices to show $\tilde{V}_2(\rho^{Pub}) < 0$. Now the equation $g^{Pub}(\rho^{Pub}) = 0$ is equivalent to $\tilde{\psi} = -\frac{1-(1+\tilde{T})\rho}{\tilde{T}\rho^2(1-\rho)}|_{\rho=\rho^{Pub}}$; using this to eliminate $\tilde{\psi}$ and simplifying, we obtain $\tilde{V}_2(\rho^{Pub}) = -\frac{[\rho(1+\tilde{T})-1]^3}{\tilde{T}\rho^2}|_{\rho=\rho^{Pub}}$, which is strictly negative as $\rho^{Pub} > \frac{1}{1+\tilde{T}}$, establishing claim (i).

Toward claim (ii), differentiate

$$\frac{d}{d\rho}V^{NF}(\rho) = \sigma_Y^2\left\{-\tilde{\psi}^2[-3\rho(1-\rho)^2 + (1-\rho)^3] + \tilde{\psi}\right\} = \sigma_Y^2\tilde{\psi}\left\{-\tilde{\psi}(1-\rho)^2(1-4\rho)+1\right\}.$$

The expression in braces is positive iff $h(\rho) := (1-\rho)^2(1-4\rho) < \frac{1}{\tilde{\psi}}$. Now for $\rho \in [0,1]$, $h(\rho)$ attains its maximum value of 1 at $\rho = 0$. Hence, if $\tilde{\psi} \le 1$, the expression is positive for all $\rho \in (0,1)$ and we conclude that $V^{NF}(\rho)$ is increasing for all $\rho \ge \rho^{Pub}$.

Combining parts (i) and (ii) yields $V^{Pub}(\rho^{Pub}) < V^{NF}(\rho^{Pub}) < V^{NF}(\rho^{NF})$ as desired. □

S.3.2 Proof of Proposition 5, Part (ii)

Since the proof follows the same steps as that of Theorem 2, we only point out the differences owing to the presence of γ_T in the terminal conditions.

After a change of variables $\tilde{v}_{6t} = \frac{v_{6t}\gamma_t}{(1-\chi_t)^2}$, $\tilde{v}_{8t} = \frac{v_{8t}\gamma_t}{1-\chi_t}$ and $\tilde{\beta}_{2t} = \frac{\beta_{2t}}{1-\chi_t}$ the boundary value problem for $(\tilde{v}_6, \tilde{v}_8, \beta_1, \tilde{\beta}_2, \beta_3, \gamma, \chi)$ is

$$\dot{\tilde{v}}_{6t} = \gamma_t\left\{\tilde{\beta}_{2t}(2\beta_{1t} + \tilde{\beta}_{2t}) + \tilde{v}_{6t}\left(\frac{\alpha_t^2}{\sigma_Y^2} + \frac{2\chi_t}{\sigma_X^2}\right)\right\} \tag{S.39}$$

$$\dot{\tilde{v}}_{8t} = \gamma_t\left\{-2\beta_{1t}(1-\beta_{3t}) - 2\tilde{\beta}_{2t} + \frac{\tilde{v}_{8t}\chi_t}{\sigma_X^2}\right\} \tag{S.40}$$

$$\dot{\beta}_{1t} = \frac{\gamma_t}{2\sigma_X^2\sigma_Y^2}\left\{2\sigma_X^2\alpha_t\beta_{1t}(\alpha_t - \beta_{1t}) - 2\sigma_Y^2\chi_t[2\tilde{\beta}_{2t}(1+\beta_{1t}\chi_t) + \beta_{1t}^2\chi_t] - \tilde{v}_{8t}\alpha_t\beta_{1t}\chi_t\right\} \tag{S.41}$$

$$\begin{aligned}\dot{\tilde{\beta}}_{2t} = \frac{\gamma_t}{2\sigma_X^2\sigma_Y^2}&\Big\{2\sigma_X^2\alpha_t(\beta_{1t}^2 + \alpha_t\tilde{\beta}_{2t}) - \alpha_t\chi_t(2\tilde{v}_{6t} + \tilde{v}_{8t}\tilde{\beta}_{2t}) \\ &+2\sigma_Y^2\tilde{\beta}_{2t}\chi_t[1-(\beta_{1t}+2\tilde{\beta}_{2t})\chi_t]\Big\}\end{aligned} \tag{S.42}$$

$$\dot{\beta}_{3t} = \frac{\gamma_t}{2\sigma_X^2\sigma_Y^2}\left\{-2\sigma_X^2\alpha_t\beta_{1t}\beta_{3t} + 2\sigma_Y^2\chi_t^2\left(2\tilde{\beta}_{2t}+\beta_{1t}\right)(1-\beta_{3t}) - \tilde{v}_{8t}\alpha_t\beta_{3t}\chi_t\right\} \tag{S.43}$$

$$\dot{\gamma}_t = -\frac{\gamma_t^2\alpha_t^2}{\sigma_Y^2} \tag{S.44}$$

$$\dot{\chi}_t = \gamma_t\left\{\frac{(1-\chi_t)\alpha_t^2}{\sigma_Y^2} - \frac{\chi_t^2}{\sigma_X^2}\right\}, \tag{S.45}$$

with boundary conditions $(v_{6T}, v_{8T}, \beta_{1T}, \beta_{2T}, \beta_{3T}) = \left(0, 0, -\frac{\psi\gamma_T}{\sigma_Y^2+\psi\gamma_T\chi_T}, 0, 1\right)$ and $(\gamma_0, \chi_0) = (\gamma^o, 0)$. We define $z, \tilde{z}, F$ and $\tilde{F}$ as in the proof of Theorem 2, but now we define $\mathbf{B} : \mathbb{R}_+^2 \to \mathbb{R}^5$ by $\mathbf{B}(\gamma, \chi) = \left(0, 0, -\frac{\psi\gamma}{\sigma_Y^2+\psi\gamma\chi}, 0, 1\right)$. Define $s_0 \in \mathbb{R}^5$ by $s_0 = \mathbf{B}(\gamma^o, 0) = (0, 0, -\psi\gamma^o/\sigma_Y^2, 0, 1)$. For all $s \in \mathcal{S}_\rho(s_0)$, define IVP-$s$ as in the proof of Theorem 2.

A solution to IVP-s solves the BVP if and only if $\tilde{z}_T(s) = \mathbf{B}(\gamma_T(s), \chi_T(s))$, which is satisfied if and only if s is a fixed point of the function $g : \mathcal{S}_\rho(s_0) \to \mathbb{R}^5$ defined by $g(s) := \mathbf{B}(\gamma_T(s), \chi_T(s)) - \int_0^T \tilde{F}(z_t(s))dt$.

Step 2 of the earlier proof then proceeds as it did before. They key difference lies in Step 3. We have $g(s) - s_0 = \Delta(s) - \int_0^T \tilde{F}(z_t(s))dt$, where

$$\begin{aligned}\Delta(s) &:= \mathbf{B}(\gamma_T(s), \chi_T(s)) - \mathbf{B}(\gamma^o, 0)\\ &= \left(0, 0, -\frac{\psi\gamma_T(s)}{\sigma_Y^2 + \psi\gamma_T(s)\chi_T(s)} + \frac{\psi\gamma^o}{\sigma_Y^2}, 0, 0\right).\end{aligned}$$

We now must bound $\Delta_3(s)$. For arbitrary $\rho, K \in \mathbb{R}_{++}^2$, define $\bar{\beta}_1(\gamma^o, \rho, K) := \rho + K + \frac{\psi\gamma^o}{\sigma_Y^2}$, $\bar{\beta}_3(\gamma^o, \rho, K) := 1 + \rho + K$ and $\bar{\alpha}(\gamma^o, \rho, K) := \bar{\beta}_1(\gamma^o, \rho, K)\bar{\chi} + \bar{\beta}_3$ which bound $|\beta_1|$, $|\beta_3|$ and $|\alpha|$, respectively. Define $\bar{\Delta}_3(\gamma^o; \rho, K) = \frac{\psi(\gamma^o)^2}{\sigma_X^2\sigma_Y^6}\left[\psi\sigma_Y^2\gamma^o\bar{\chi}^2 + \bar{\alpha}^2\sigma_X^2(\sigma_Y^2 + \psi\gamma^o)\right]$, where the dependence on ρ, K is through $\bar{\alpha}$. By the fundamental theorem of calculus, we have $\Delta_3(s) = \int_0^T \frac{d}{dt}\mathbf{B}_3(\gamma_t(s), \chi_t(s))dt$, where

$$\frac{d}{dt}\mathbf{B}_3(\gamma_t(s), \chi_t(s)) = \frac{\psi\gamma_t^2\left[-\psi\sigma_Y^2\gamma_T\chi_t^2 + \alpha_t^2\sigma_X^2(\sigma_Y^2 + \psi\gamma_t(1-\chi_t))\right]}{\sigma_X^2\sigma_Y^2(\sigma_Y^2 + \psi\gamma_t\chi_t)^2}.$$

By construction, $|\frac{d}{dt}\mathbf{B}_3(\gamma_t(s), \chi_t(s))| \leq \bar{\Delta}_3(\gamma^o; \rho, K)$, and thus $|\Delta_3(s)| \leq T\bar{\Delta}_3(\gamma^o; \rho, K)$. Note that $\bar{\Delta}_3(\gamma^o; \rho, K)$ is proportional to γ^o. For completeness, define the vector $\bar{\Delta}(\gamma^o; \rho, K)$ by specifying $\bar{\Delta}_i(\gamma^o; \rho, K) = 0$ for $i \in \{1, 2, 4, 5\}$, and observe that $\Delta_i(s) = T\bar{\Delta}_i(\gamma^o; \rho, K) = 0$ for $i \in \{1, 2, 4, 5\}$. The threshold $T(\gamma^o; \rho, K)$ can now be defined as in the proof of Theorem 2, and the remaining steps of the proof proceed analogously.

Using the solution to the BVP and the facts above, we solve for the rest of the equilibrium coefficients as outlined in Section B.3. We conclude that a linear Markov equilibrium exists.

S.3.3 Proof of Proposition 6

The proof is by contradiction; suppose that a linear Markov equilibrium exists in which the long-run player plays $a_t = \beta_{0t} + \beta_{1t} M_t + \beta_{2t} L_t + \beta_{3t}\theta$, where the β_i are differentiable. Define $\alpha_{0t} := \beta_{0t}$, $\alpha_{2t} := \beta_{2t} + \beta_{1t}(1-\chi_t)$ and $\alpha_{3t} = \beta_{3t} + \beta_{1t}\chi_t$. We derive a collection of necessary conditions for equilibrium and show that there is no real value of α_{30} for which they can be satisfied.

From the long-run player's perspective, given an action profile $(a_t)_{t\geq 0}$, the state variables M_t and L_t evolve according to $dL_t = \mu_L(a_t)dt + \sigma_L dZ_t^X$ and $dM_t = \mu_M(a_t)dt + \sigma_M dZ_t^X$, where $\sigma_L = \frac{\hat{B}_t\sigma_X}{1-\chi_t}$, $\sigma_M = \hat{B}_t\sigma_X$, $\hat{B}_t = \frac{\gamma_t(\alpha_{3t}+\xi\chi_t)}{\sigma_X^2}$, $\Sigma = \frac{1}{\sigma_X^2} + \frac{1}{\sigma_Y^2}$, $\mu_L(a) = \frac{\hat{B}_t}{1-\chi_t}[a + \xi(m-l) - \alpha_{0t} - (\alpha_{2t}+\alpha_{3t})l]$, and $\mu_M(a) = \Sigma\alpha_{3t}\gamma_t\,[a - \alpha_{0t} - \alpha_{2t}l - \alpha_{3t}m]$, and where

$$\dot{\gamma}_t = -\gamma_t^2\alpha_{3t}^2\Sigma \tag{S.46}$$

$$\dot{\chi}_t = \gamma_t\alpha_{3t}^2\Sigma(1-\chi_t) - (\alpha_{3t}+\xi\chi_t)\hat{B}_t. \tag{S.47}$$

In any conjectured LME, for $(\theta, M_t, L_t, t) = (\theta, m, l, t)$, the long-run player's value function is of the form $V(\theta,m,l,t) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}l + v_{4t}\theta^2 + v_{5t}m^2 + v_{6t}l^2 + v_{7t}\theta m + v_{8t}\theta l + v_{9t}ml$. Using Itô's lemma, for $t \in [0,T)$, this must satisfy the HJB equation

$$0 = \sup_a \left\{(\theta - L)a + V_t + \mu_L(a)V_\ell + \mu_M(a)V_m + \frac{1}{2}\sigma_M^2 V_{mm} + \frac{1}{2}\sigma_L^2 V_{\ell\ell} + \sigma_L\sigma_M V_{\ell m}\right\},$$

and by linearity in a, existence of an LME requires the indifference condition

$$0 = (\theta - l) + \frac{\hat{B}_t}{1-\chi_t}V_\ell + \alpha_{3t}\gamma_t\Sigma V_m. \tag{S.48}$$

Since (S.48) must hold for all values of θ, m and l, we match coefficients to obtain:

$$\text{constant}: \qquad 0 = \gamma_t\left[\Sigma v_{2t}\alpha_{3t} + \frac{v_{3t}(\alpha_{3t}+\xi\chi_t)}{\sigma_X^2(1-\chi_t)}\right] \tag{S.49}$$

$$\theta: \qquad 0 = 1 + \Sigma v_{7t}\alpha_{3t}\gamma_t + \frac{v_{8t}\gamma_t(\alpha_{3t}+\xi\chi_t)}{\sigma_X^2(1-\chi_t)} \tag{S.50}$$

$$m: \qquad 0 = \gamma_t\left[2\Sigma v_{5t}\alpha_{3t} + \frac{v_{9t}(\alpha_{3t}+\xi\chi_t)}{\sigma_X^2(1-\chi_t)}\right] \tag{S.51}$$

$$l: \qquad 0 = -1 + \Sigma v_{9t}\alpha_{3t}\gamma_t + \frac{2v_{6t}\gamma_t(\alpha_{3t}+\xi\chi_t)}{\sigma_X^2(1-\chi_t)}. \tag{S.52}$$

Note that $\gamma_0 = \gamma^o > 0$, and since $\chi_0 = 0$, (S.50) (or (S.52)) implies $\alpha_{30} \neq 0$. Hence, for sufficiently small t, we can solve (S.51)-(S.52) to obtain

$$v_{5t} = -\frac{\sigma_Y^2 v_{9t}(\alpha_{3t} + \xi\chi_t)}{2(\sigma_X^2 + \sigma_Y^2)\alpha_{3t}(1-\chi_t)} \tag{S.53}$$

$$v_{6t} = \frac{[\sigma_X^2\sigma_Y^2 - (\sigma_X^2 + \sigma_Y^2)v_{9t}\alpha_{3t}\gamma_t](1-\chi_t)}{2\sigma_Y^2\gamma_t(\alpha_{3t} + \xi\chi_t)}. \tag{S.54}$$

Differentiate v_{5t}, use (S.47) to replace $\dot{\chi}_t$ and evaluate at time $t = 0$ to obtain

$$\dot{v}_{50} = -\frac{v_{90}\alpha_{30}(\xi + \alpha_{30})\gamma_0 + \sigma_Y^2\dot{v}_{90}}{2(\sigma_X^2 + \sigma_Y^2)}. \tag{S.55}$$

On the other hand, the indifference condition reduces the HJB equation to

$$0 = V_t + \mu_L(0)V_\ell + \mu_M(0)V_m + \frac{1}{2}\sigma_M^2 V_{mm} + \frac{1}{2}\sigma_L^2 V_{\ell\ell} + \sigma_L\sigma_M V_{\ell m}. \tag{S.56}$$

Now (S.56) must hold for all θ, m and l, and by matching coefficients, we obtain a set of 10 equations. In particular, the coefficients on m^2 and ml at $t = 0$ must vanish; using (S.53), (S.54) and (S.55) to eliminate v_5, v_6 and $\dot{v}_5$, and solving for $\dot{v}_{90}$, we obtain

$$\dot{v}_{90} = \frac{(\sigma_X^2 + 2\sigma_Y^2)v_{90}\alpha_{30}(\xi + \alpha_{30})\gamma_0}{\sigma_X^2\sigma_Y^2} \text{ and } \dot{v}_{90} = -\xi + \frac{(\sigma_X^2 + 2\sigma_Y^2)v_{90}\alpha_{30}(\xi + \alpha_{30})\gamma_0}{\sigma_X^2\sigma_Y^2},$$

which gives us the desired contradiction. (Note that if $v_{90} \neq 0$, equality (of the form $+\infty = +\infty$) between the right hand sides is achieved if $\alpha_{30} = +\infty$, supporting the interpretation that the long-run player would trade away all information in the first instant.) □

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