Dynamic Oligopoly with Incomplete Information

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Motivation

- New markets: firms often know their costs better than rivals $\Rightarrow$ opportunities for **signaling** and **learning**
- How does "jockeying for a position" play out?
  - Strategies; learning; prices, profits and welfare
- Challenge: handling beliefs $\Rightarrow$ existing works typically assume:
  - two periods or static:
    - one-sided info: Milgrom and Roberts 82, Fudenberg and Tirole 86, Mailath 89
    - two-sided info: Bernhardt and Taub 15 (supply functions)
  - patient firms (e.g., Athey & Bagwell 08), or
  - non-standard solution concepts (e.g., Fershtman & Pakes, 12; Doraszelski, Lewis, Pakes, 14)
- This paper: a tractable analysis of **dynamic oligopolistic competition** with incomplete information

Model

"Repeated Cournot w/ pvte. costs and imperfect monitoring"

- Time is continuous $t \in [0, T]$
- n firms; compete in quantities $Q_t^i \in \mathbb{R}$, $i = 1, ..., n$
- **Incomplete information:** costs are private; $C^i \overset{\text{iid}}{\sim} \mathcal{N}(\pi_0, g_0)$
- **Imperfect monitoring:** firms only observe cumulative price process:

$$dY_t = \left(\bar{p} - \sum_i Q_t^i\right) dt + \sigma \, dZ_t$$

  where $(Z_t)_{t \geq 0}$ is a Brownian motion

- Firm i's flow payoff: $Q_t^i(dY_t - C^i dt)$; Total expected payoff:

$$\mathbb{E}\left[\int_0^T e^{-rt}\left(\bar{p} - \sum_j Q_t^j - C^i\right) Q_t^i dt\right]$$

Preview

- **Markov strategies**: condition only on beliefs about costs and calendar time
- Representation of symmetric **linear Markov** strategies as

$$
\begin{aligned}
Q_t^i &= \alpha_t C^i + \beta_t \Pi_t + \delta_t, \\
\text{where } \mathit{\Pi}_t &:= \mathbb{E}\left[\frac{1}{n}\Sigma_j C^j \mid Y^t\right] \quad \text{“public belief”}
\end{aligned}
$$

- Characterization of symmetric linear Markov equilibria: system of nonlinear ODEs
- Economic implications

Preview

1. Limit behavior as $T \to \infty$:
   - Play converges to static complete info Nash; firm only learns avg. cost
2. Incentives: signal jamming, learning and signaling
   - Non-monotone α and β decreasing
3. Implications for prices, quantities:
   - Prices have an "upward trend"; difference $Q^i - Q^j$ is non-monotone
4. Welfare effects of dynamic competition:
   - Consumer surplus higher than static case
   - Expected profits may be highest in the medium run

Strategies and Equilibrium

- Info comes from (C, Y) only; Y satisfies full-support assumption
  - "Nash eq. is outcome-equivalent to sequential eq." $\Rightarrow$ focus on Nash
- A profile $(Q_t^1, ..., Q_t^n)_{t\in[0,T]}$ is *admissible* if

(i) $(Q_t^i)_{t\in[0,T]}$ is $\mathcal{F}_t^{C^i,Y}-$measurable

(ii) $\mathbb{E}[\int_0^T (Q_t^i)^2 dt] < \infty$

Definition (Pure-Strategy Nash Equilibrium)

Profile Q such that no firm can improve time-zero expected profits by deviating to an admissible $\tilde{Q}^i$

Linear Strategies

- Tractable **learning** dynamics $\Rightarrow$ linearity in C

- A strategy is said to be *linear* if

$$Q_t^i \quad = \quad \alpha_t C^i + \int_0^t f_s^t dY_s + \delta_t$$

where $\alpha, \delta : [0, T] \to \mathbb{R}$ and $f^t : [0, t] \to \mathbb{R},\ t \in [0, T]$.

- Focus on **symmetric linear** equilibria (α, f, δ)

Beliefs and Linear Markov Strategies

Private Learning

- Fix strategy profile $Q^{-i} = (\alpha, f, \delta)$. Firm i always observes

$$dY_t^i \;:=\; dY_t - \underbrace{\left[\bar{p} - (n-1)\left(\int_0^t f_s^t\, dY_s + \delta_t\right)\right]}_{\text{public component of rival's strategies}} dt - \underbrace{Q_t^i\, dt}_{\text{past play}}$$

$$\Rightarrow dY_t^i \;=\; -\alpha \sum_{j \neq i} C^j\, dt + \sigma\, dZ_t \;:\; \textbf{pvte. signal of rivals' costs}$$

- By symmetry and normality, firm i's belief is summarized by

$$M_t^i = \frac{1}{n-1}\mathbb{E}_t^i\Big[\sum_{j \neq i} C^j\Big] \quad \text{and} \quad \gamma_t^M = \frac{(n-1)g_0}{1 + (n-1)g_0 \int_0^t (\frac{\alpha_s}{\sigma})^2\, ds}.$$

with M_t^i is private and γ^M is deterministic (given α)

- Conditioning actions of $M^i \Rightarrow$ Forecasting the forecasts of others

Representation of M^i Through Public Belief

- Fix strategy profile (α, f, δ). Define the **public belief** $\sum_j C^j \big| Y$

$$\mathit{\Pi}_t := \frac{1}{n}\mathbb{E}\Big[\sum_j C^j\Big] \text{ and } \gamma_t := \frac{ng_0}{1 + ng_0 \int_0^t \left(\frac{\alpha_s}{\sigma}\right)^2 ds}$$

- Belief of an outsider (cf. market maker in finance models)

Lemma (Adaptation of Foster and Viswanathan (JF, 1996))

Under a symmetric linear strategy profile, for each firm i,

$$M_t^i = z_t \mathit{\Pi}_t - (z_t - 1) C^i,\ t \geq 0, \quad \textit{where}$$

$$z_t = \frac{n\gamma_t^M}{(n-1)\gamma_t} = \frac{n^2 g_0}{n(n-1)g_0 + \gamma_t} \in \left[1, \frac{n}{n-1}\right]$$

- Along the path of play, payoff-relevant info summarized by (C, Π, γ)

Linear Markov Strategies

- A strategy is **Markov** if each Q^i depends on history only through the firm i's beliefs about $(C^1, ..., C^n)$ and calendar time
  - (C, Π, γ, t) is sufficient
- Gaussian learning: $\mathit{\Pi}_t = \mathit{\Pi}_t[Y]$ is linear and γ deterministic

Lemma

*A symmetric linear strategy profile is **Markov** if and only if there exist coefficients with $\alpha, \beta, \delta : [0, T] \to \mathbb{R}$ such that*

$$\boxed{Q_t = \alpha_t C + \beta_t \mathit{\Pi}_t + \delta_t}$$

- Focus on **symmetric linear Markov** equilibrium
- Performing eq. analysis requires evaluating hypothetical deviations

Beliefs Off the Equilibrium Path

- Deviations $\Rightarrow \Pi_t$ is biased and Lemma no longer holds
- But $M_t^i = M_t^i[Y^i]$ is uniquely determined by $(Y_s^i : 0 \le s \le t)$
- Construct **counterfactual** public belief: $\hat{\Pi}_t = \hat{\Pi}_t[Y^i]$

Lemma

$M_t^i = z_t \hat{\Pi}_t - (z_t - 1) C^i$ *a.s.,* $t \in [0, T]$. *If all firms play symmetrically, then* $\hat{\Pi} = \Pi$

- If other firms play a symmetric Markov profile, then firm i's payoff-relevant *state* is $(C^i, \Pi_t, \hat{\Pi}_t^i, t)$

Dynamic Programming

Best-Response Problem

- Fixing strategy profile $Q^{-i} = (\alpha, \beta, \delta)$, firm i solves:

$$\max_{(Q^i_s)_{0\le s\le t}} \mathbb{E}\int_0^T e^{-rs}\big[\bar{p} - Q^i_s - \underbrace{(n-1)(\alpha_s M^i_s + \beta_s \Pi_s + \delta_s)}_{\sum_{j\neq i}\mathbb{E}[Q^j_s|\mathcal{F}^i_s]} - c\big]\, Q^i_s ds$$

s.t. $d\Pi_s = \lambda_s\{[(\alpha_s + \beta_s)\Pi_s + \delta_s - Q^i_s] + (n-1)\alpha_s[\Pi_s - M^i_s]\}ds + \lambda_s\sigma\, dZ^i_s,$

$$d\hat{\Pi}^i_s = \lambda_s[\alpha_s(\hat{\Pi}^i_s - c) + (n-1)\alpha_s(\hat{\Pi}^i_s - M^i_s)]ds + \lambda_s\sigma\, dZ^i_s,$$

$$M^i_s = z_s\hat{\Pi}^i_s + (1-z_s)c, \quad \lambda_t := -\frac{\alpha_t\gamma_t}{n\sigma^2}, \quad \Pi_t = \pi, \quad \hat{\Pi}^i_t = \hat{\pi},$$

where $(Q^i_s)_{0\le s\le T}$ is p.m. w.r.t. $\mathcal{F}^{C^i,\Pi,Z^i_t}_t$, $t \ge 0$

- B-R problem is a L-Q control problem: quadratic value and linear BR

Finding Markov Equilibria

- Suppose that (α, β, δ) are such that there exists $V(c, \pi, \hat{\pi}, t)$ solving

$$
\begin{aligned}
rV(c,\pi,\hat{\pi},t) \quad &= \quad \sup_{q\in\mathbb{R}} \Bigg\{ \Big[\bar{p} - q - (n-1)\overbrace{\big(\alpha_t(z_t\hat{\pi} + (1-z_t)c) + \beta_t\pi + \delta_t\big)}^{\mathbb{E}[Q_t^j|\mathcal{F}_t^i]} - c\Big] q \\
&\qquad + \mathcal{L}^q V(c,\pi,\hat{\pi},t) \Bigg\} \\
s.t. \quad & \quad \arg\max \ \text{RHS}(c,\pi,\hat{\pi},t)|_{\pi=\hat{\pi}} = \alpha_t c + \beta_t \pi + \delta_t
\end{aligned}
$$

with $\mathcal{L}^q V = \mu_t(q)\frac{\partial V}{\partial \pi} + \hat{\mu}_t \frac{\partial V}{\partial \hat{\pi}} + \frac{\partial V}{\partial t} + \frac{\lambda_t^2 \sigma^2}{2}\left(\frac{\partial^2 V}{\partial \pi^2} + 2\frac{\partial^2 V}{\partial \pi \partial \hat{\pi}} + \frac{\partial^2 V}{\partial \hat{\pi}^2}\right)$

$\Rightarrow (\alpha, \beta, \delta)$ is a **Markov eq.**, and $V(c, \pi, \pi, t)$ the on-path payoff

- Quadratic solution in $(c, \pi, \hat{\pi})$ with time-dependent coefficients
  - ODEs are non-standard: game is not a L-Q game; $(\gamma)_{t\in[0,T]}$ is non-linear

Characterization: Boundary Value Problem

Let $\alpha^m(x)$, $\beta^m(x)$ and $\xi^m(x)$: myopic best-response given variance $x > 0$ ▸ Myopic coefficients. Consider the BVP

$$\dot{\alpha}_t = r\alpha_t \frac{\alpha_t - \alpha^m(\gamma_t)}{\alpha^m(\gamma_t)} - \frac{\alpha_t^2 \beta_t \gamma_t [(n-1)n\alpha_t (z_t - 1) + 1]}{n\sigma^2} \tag{1}$$

$$\dot{\beta}_t = r\alpha_t \frac{(n+1)(\beta^m(\gamma_t) - \beta_t)}{(n+1)\beta^m(\gamma_t) + 1} + \frac{\alpha_t \beta_t \gamma_t \left[n\alpha_t (n + 1 - (n-1)z_t - (n^2 - 1)\beta_t (z_t - 1)) + (n-1)\beta_t \right]}{n(n+1)\sigma^2}, \tag{2}$$

$$\dot{\gamma}_t = -\frac{\alpha_t^2 \gamma_t^2}{\sigma^2}, \tag{3}$$

$$\dot{\xi}_t = r\alpha_t \frac{(n+1)(\xi^m(\gamma_t) - \xi_t)}{2\xi^m(\gamma_t) + 1} + \frac{\alpha_t \gamma_t \xi_t}{n\sigma^2} \left[\xi_t - (n\alpha_t((n-1)\beta_t(z_t - 1) - 1) + \beta_t) \right] - \frac{(n-1)\alpha_t^2 \beta_t \gamma_t z_t}{2\sigma^2} \tag{4}$$

with terminal conditions $\alpha_T = \alpha^m(\gamma_T), \beta_T = \beta^m(\gamma_T)$ and $\xi_T = \xi^m(\gamma_T)$, and initial condition $\gamma_0 = ng_0$, respectively

Existence and Characterization of Linear Markov Eq.

Theorem

A symmetric linear Markov equilibrium with differentiable coefficients (α, β, δ) exists if and only if

(i) $\delta = -\bar{p}(\alpha + \beta)$

(ii) $\exists\, \xi : [0, T] \to \mathbb{R}$ *such that* $(\alpha, \beta, \gamma, \xi)$ *solve the BVP (1)-(4).*

In any such an equilibrium (on- and off-path) behavior is given by

$$\boxed{Q^*(c, \pi, \hat{\pi}, t) = \alpha_t c + \beta_t \pi + \delta_t + \xi_t(\hat{\pi} - \pi)}$$

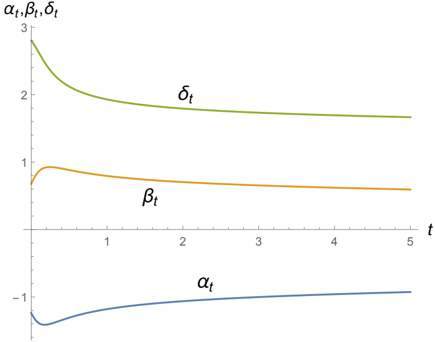
A sufficient condition for a solution to the BVP to exist is

$$\frac{n g_0}{\sigma^2} < \max\left\{\frac{4r}{27}, \frac{1}{3T}\right\}.$$

Equilibrium Coefficients: $Q_t^i = \alpha_t C^i + \beta_t \Pi_t + \delta_t$

Proposition

1. *α is $\searrow$ and β is $\nearrow$ at zero; for T large, α is $\nearrow$ and β $\searrow$ at T*
2. *δ is eventually decreasing*
3. *$r = 0$: α is quasi-convex; β is quasi-concave; δ is strictly decreasing.*



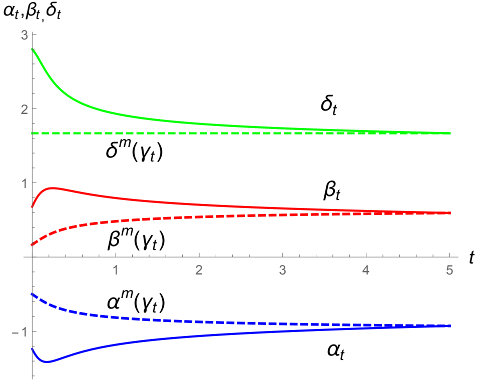
Economic Implications

1. Beliefs and behavior as $T \to \infty$
2. Incentives: signal jamming, learning and signaling
3. Paths of prices and quantities
4. Expected flow profits and consumer surplus

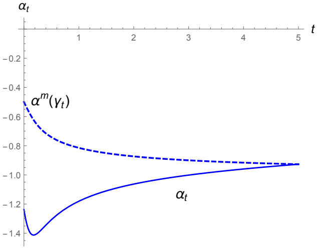
Preliminary Result: Dynamic vs. Myopic Coefficients

Proposition

$(-\alpha_t, \beta_t, \delta_t) \geq (-\alpha^m(\gamma_t), \beta^m(\gamma_t), \delta^m(\gamma_t))$



1) Learning and Limit Behavior



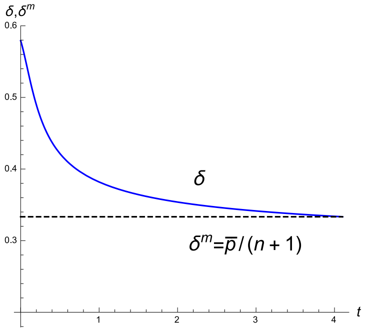
$\gamma_t \propto (1 + ng_0 \int_0^T \frac{\alpha_s^2}{\sigma^2} ds)^{-1}$; $q^N(\vec{C})$: Nash eq. of static complete info

Corollary

$\gamma_t^T \to 0$ *as* t *grows, uniformly along any sequence of games. Thus, for all* $\epsilon > 0$*, there exists* $t_\epsilon < \infty$ *s.t. in every symmetric linear Markov eq.*

$$\mathbb{P}[\| Q_t - q^N(\vec{C})\| \leq \epsilon] > 1 - \epsilon \ \ \textit{for} \ \ t \in [t_\epsilon, T]$$

2) Incentives: Signal-Jamming δ



Corollary (Overproduction)

$$\mathbb{E}\left[\sum_i Q_t^i\right] = n\delta_t\left(1 - \frac{\pi_0}{\bar{p}}\right) > \frac{n}{n+1}(\bar{p} - \pi_0) \text{ if } \bar{p} - \pi_0 > 0.$$

2) Incentives: Learning and Signaling

- Goal: understand forces behind non-monotonicities
- Learning $\Rightarrow$ move from $(C^i, \mathit{\Pi})$ to (C^i, M^i) space:

$$Q_t^i = \alpha_t C^i + \beta_t \Pi_t + \delta_t = \underbrace{\left(\alpha_t - \frac{1-z_t}{z_t}\beta_t\right)}_{\hat{\alpha}_t :=} C^i + \underbrace{\frac{\beta}{z_t}}_{\hat{\beta}_t :=} M_t^i + \delta_t$$

- **Learning:** $(\hat{\alpha}_t^{m,BR}, \hat{\beta}_t^{m,BR}, \hat{\delta}_t^{m,BR})$
  - Adjusting output as information accumulates
- **Signaling:** $(\hat{\alpha}_t - \hat{\alpha}_t^{m,BR}, \hat{\beta}_t - \hat{\beta}_t^{m,BR}, \hat{\delta}_t - \hat{\delta}_t^{m,BR})$
  - Deviations from myopic production: affecting continuation values

2) Incentives: Learning and Signaling

Proposition

$\hat{\alpha}_t - \hat{\alpha}_t^{m,BR} < 0$, $\hat{\beta}_t - \hat{\beta}_t^{m,BR} > 0$ and $\hat{\delta}_t - \hat{\delta}_t^{m,BR} > 0$. If $r = 0$, $|\hat{\alpha}_t - \hat{\alpha}_t^{m,BR}|$, $|\hat{\beta}_t - \hat{\beta}_t^{m,BR}|$ and $|\hat{\delta}_t - \hat{\delta}_t^{m,BR}|$ are decreasing

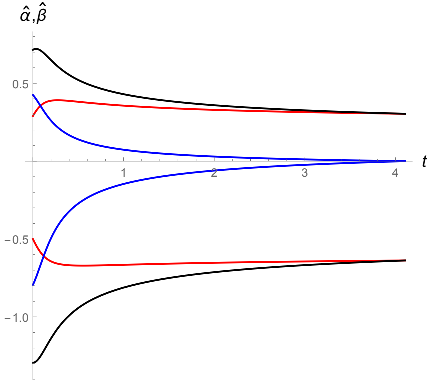


Figure : $(\hat{\alpha}_t, \hat{\beta}_t)$; $(\hat{\alpha}_t^{m,BR}, \hat{\beta}_t^{m,BR})$; $(\hat{\alpha}_t - \hat{\alpha}_t^{m,BR}, \hat{\beta}_t - \hat{\beta}_t^{m,BR})$

Incentives: Summary

- Fix t. Behavior across different types:

$$Q_t^i - Q_t^{i,m,BR} = \underbrace{\hat{\delta}_t - \hat{\delta}_t^{m,BR}}_{>0} \qquad \text{(all firms have the incentive to overproduce)}$$

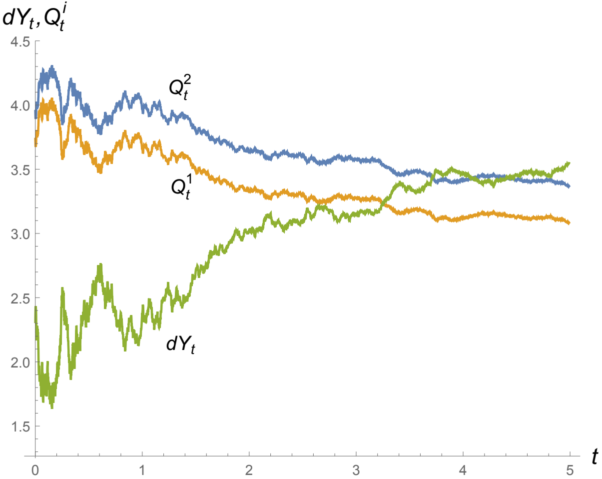
$$+ \underbrace{\left(\hat{\alpha}_t - \hat{\alpha}_t^{m,BR}\right)}_{<0} C^i \qquad \text{(more efficient firms scale back less)}$$

$$+ \underbrace{\left(\hat{\beta}_t - \hat{\beta}_t^{m,BR}\right)}_{>0} M_t^i \qquad \text{(optimistic firms produce more)}$$

- Fix type. Using $\delta = -\bar{p}(\alpha + \beta)$, behavior across time:

$$Q_t^i - Q_t^{i,m,BR} = -(\hat{\alpha}_t - \hat{\alpha}_t^{m,BR})[\bar{p} - C^i] - (\hat{\beta}_t - \hat{\beta}_t^{m,BR})[\bar{p} - M_t^i]$$

3) Prices and Quantities



3) Prices and Quantities

- Over-production $\Rightarrow$ in expectation prices everywhere depressed
  - Expected prices eventually increasing in t
- If $r = 0$, expected total quantity decreases (price increases) in t
- Output differences across firms are deterministic (given C):

$$Q_t^i - Q_t^j = \alpha_t (C^i - C^j),$$

where α is non-monotone (for large T) and "U-shaped" (if $r = 0$)

4) Welfare Effects

- Expected flow consumer surplus $>$ complete-information, for all t
- Expected flow profits $<$ complete-information at $t \in \{0, T\}$
- Non-monotone $\alpha \Rightarrow$ implications for allocative efficiency:
  - Low-cost firm has largest advantage in the medium run
  - "Taking turns" from ex-ante perspective
  - Variance increases expected profits (esp. if avg. margin is low)
  - Highest industry profits at interior t

Expected Flow Profits

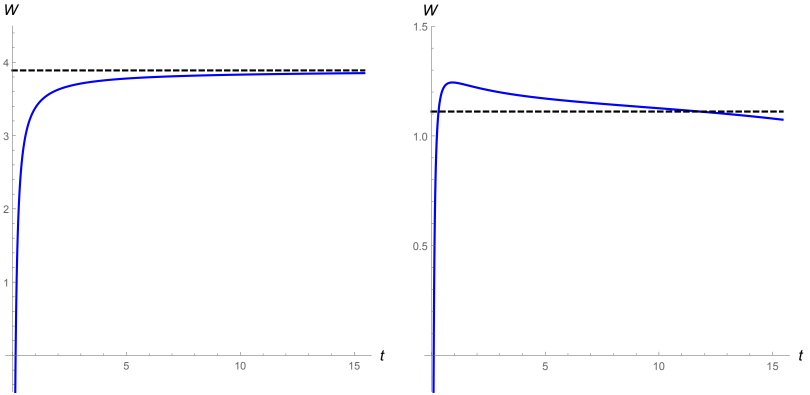


Figure : Comparison of expected flow profits under complete information (dashed) and incomplete information (blue) for parameter values $\pi_0 = 0$ (left), $\pi_0 = \bar{p}$ (right), and $(r, \sigma, n, \bar{p}, T, g_0) = (0.2, 1, 2, 5, 15.44, 2)$.

Infinite Horizon

- Sequence of symmetric linear Markov equilibria indexed by $T^n \nearrow \infty$
- Let $g_0/\sigma^2 < 4r/(27n)$.

Proposition

Any sequence of symmetric linear Markov equilibria contains a subsequence that converges uniformly to a symmetric linear Markov equilibrium $(\alpha^*, \beta^*, \delta^*, \xi^*, \gamma^*)$ *of the infinite-horizon game.*

Furthermore, $\delta^* = -\bar{p}(\alpha^* + \beta^*)$ *and* $(\alpha^*, \beta^*, \xi^*, \gamma^*)$ *is a solution to the BVP on* $[0, \infty)$ *with* $\lim_{t\to\infty} \alpha_t^* = \alpha^m(0)$, $\lim_{t\to\infty} \beta_t^* = \beta^m(0)$, $\lim_{t\to\infty} \xi_t^* = \xi^m(0)$, *and* $\gamma_0^* = ng_0$.

Literature

- Industrial Organization
  - Milgrom and Roberts 82, Fudenberg and Tirole 86, Mailath 89, etc.
  - Raith 96, Vives 08 book, Bernhardt and Taub 15
  - Fershtman and Pakes 12; Doraszelski, Lewis and Pakes 14
- Finance
  - Kyle 85; Foster and Viswanathan 96; Back, Cao and Willard 00
- Repeated Bayesian Games
  - Athey and Bagwell 08
  - Aumann and Maschler 95, Peski 14
  - Horner and Lovo 09
  - Escobar and Toikka 13; Horner, Takahashi and Vieille 15
- Continuous-time Games and Private Beliefs
  - Cisternas 15

Conclusions

- A tractable dynamic oligopoly game with independent private values and imperfect monitoring
  - Symmetric linear Markov equilibrium in private and public info
  - Private information is gradually revealed over time; convergence to full info play
  - Overproduction $\Rightarrow$ depressed prices
  - Non-monotone incentives
    - Non-monotone differences in quantities: $Q_t^i - Q_t^j = \alpha_t(C^j - C^j)$
    - Non-monotone ex-ante flow profits
- Extensions
  - Correlation
  - Interdependent values
  - Asymmetric case
  - Changing costs

Myopic Coefficients

Given $\gamma_t = x \in \mathbb{R}_+$, the unique equilibrium for myopic firms is given by

$$\alpha^m(x) = -\frac{(n-1)ng_0 + x}{(n-1)ng_0 + (n+1)x} \qquad < -\frac{1}{2},$$

$$\beta^m(x) = \frac{(n-1)n^2 g_0}{(n+1)[(n-1)ng_0 + (n+1)x]} \qquad > \frac{n-1}{2(n+1)},$$

$$\delta^m(x) = -\bar{p}(\alpha^m(x) + \beta^m(x)) \qquad = \frac{\bar{p}}{n+1}$$

$$\xi^m(x) = \frac{(n-1)n^2 g_0}{2[(n-1)ng_0 + (n+1)x]} \qquad > \frac{n-1}{2}.$$

▸ Boundary Value Problem