Lecture 1: Probability Basics

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Probability Space

- Ω a set of states of the world (not nec. finite)
- A collection $\mathcal{F}$ of subsets of Ω is a σ-**algebra**, if:

  (i) $\emptyset \in \mathcal{F}$

  (ii) If $B \in \mathcal{F}$, then $B^c \in \mathcal{F}$

  (iii) If $B_i \in \mathcal{F}$ for $i = 1, 2, ...$, then $\bigcup_{i \in \mathbb{N}} B_i \in \mathcal{F}$.
- The *Borelians* in $\mathbb{R}^n$ is the smallest $\sigma-$algebra that contains the all open balls. Denoted $\mathcal{B}(\mathbb{R}^n)$
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a **probability measure** if $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$ and if $\{B_i | i \in I\}$ is a countable disjoint family of sets, then $\mathbb{P}\left(\bigcup_{i \in I} B_i\right) = \sum_{i \in I} \mathbb{P}(B_i)$
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a **Probability space** – our model for uncertainty.

Random Variables and Expectations

- $X : \Omega \to \mathbb{R}$ is a random variable if X is $\mathcal{F}$-measurable, that is, for every Borelian $A \subset \mathbb{R}$, $X^{-1}(A) \in \mathcal{F}$
- $\sigma(X) :=$ the smallest $\sigma-$algebra that makes a r.v. X-measurable; i.e., **the information generated by** X
- A random variable is **simple** if it has the form $X = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}$, $\alpha_i \in \mathbb{R}$, $B_i \in \mathcal{F}$, $n \in \mathbb{N}$ ($\mathcal{F}$-measurable by construction)
- For a simple random variable, set $\mathbb{E}[X] := \sum_{i=1}^{n} \alpha_i \mathbb{P}(B_i)$
- If $X \geq 0$, $\mathcal{F}-$measurable, $\mathbb{E}[X] = \sup_{Y \leq X, \text{ Y simple}} \mathbb{E}[Y]$
- For any r.v. X, define $\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$ if at least one term is finite. **Obs:** $X^+ := \max(X, 0)$ and $X^- := \min(X, 0)$

Conditional Expectations

- Let $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra (intuitively, $\mathcal{F}$ has more (i.e., finer) information than $\mathcal{G}$) and X a r.v. in $(\Omega, \mathcal{F}, \mathbb{P})$

- The **conditional expectation** of X w.r.t. $\mathcal{G}$, denoted $\mathbb{E}[X|\mathcal{G}]$, is any random variable Y in $(\Omega, \mathcal{G}, \mathbb{P})$ (i.e., Y is $\mathcal{G}$-measurable) such that $\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A]$ for all $A \in \mathcal{G}$.
  Obs: $\mathbb{1}_A(\omega) = 1$ if $\omega \in A$, and zero otherwise

- Notion equivalent to: Y r.v. such that $\mathbb{E}[XZ] = \mathbb{E}[YZ]$ for all $\mathcal{G}$-measurable Z s.t. $\mathbb{E}[XZ]$ is well-defined;

- **Radon-Nikodym theorem** states that conditional expectations exist and are "unique" (up to a set of states of the world of measure zero– "almost surely", or, "a.s.").

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Stochastic Processes

- Let $\mathbb{T} = [0, T]$ or $[0, \infty)$ and $(\Omega, \mathcal{F}, \mathbb{P})$
- A **stochastic process** is a family of random variables indexed by time. We write $X := (X_t)_{t \in \mathbb{T}}$ (i.e. for each t, $X_t : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable)
  - "A random function that unfolds over time"
- For each $\omega \in \Omega$, the function $X(\omega) : t \in \mathbb{T} \mapsto X_t(\omega)$ is called a **sample path** (or realization path)
- Examples:
  1. $X_s = 0$ for $s \in [0, 1)$. For $s > 1$, $X_s = s$ w/prob 1/2 and $X_s = -1/2$ w/prob 1/2
  2. A jump τ is distributed on $[0, \infty)$ with density $\kappa e^{-\kappa t}$. Let $X_s = 0$ for $s < \tau$, and $X_s = 1$ for $s \geq \tau$. Let $Y_s = -\kappa s$ for $s < \tau$ and $Y_s = 1 - \kappa\tau$ for $s > \tau$. Plot some sample paths of the previous stochastic processes

Filtrations and Adaptability

- A **filtration** is a collection of $\sigma-$algebras $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{T}}$, s.t. $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for $t < s$
- Let $X := (X_t)_{t\in\mathbb{T}}$ a stochastic process, the **natural filtration** associated with X is $\mathcal{F}_t^X := \sigma\left(X_s : 0 \leq s \leq t\right)$
- In the sequel we fix a **filtered probability space** $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$
- A stochastic process $X := (X_t)_{t\in\mathbb{T}}$ is **adapted** if each X_t is $\mathcal{F}_t$-measurable (i.e. $\sigma(X_t) \subset \mathcal{F}_t$)
- Example: In example [2] in the previous slide, are the filtrations $(\mathcal{F}_t^X)_{t\geq 0}$ and $(\mathcal{F}_t^Y)_{t\geq 0}$ the same or different?

Martingales

Definition

An adapted process $X := (X_t)_{t\in\mathbb{T}}$ is a **martingale** if $\mathbb{E}[|X_t|] < \infty$ for all t and

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s,\ s \leq t.$$

An adapted process $X := (X_t)_{t\in\mathbb{T}}$ is a $\mathbb{F}$-**supermartingale** if $\mathbb{E}[\min(X_t, 0)] < \infty$ for all $t \in \mathbb{T}$ and $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$ a.s. for all $0 \leq s \leq t,\ s, t \in \mathbb{T}$. X is a **submartingale** if $-X$ is a supermartingale ($\geq$ in the last inequality).

Example: Let ξ a random variable on $(\Omega, \mathcal{F})$ with $\mathbb{E}[|\xi|] < \infty$. Then the process defined by

$$X_t = \mathbb{E}[\xi|\mathcal{F}_t],\ t \in \mathbb{T},$$

is a martingale (LIE).

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Brownian Motion

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. A stochastic process $B := (B_t)_{t\geq 0}$ is a **Brownian Motion** if:

(i) $B_0 = 0$ a.s.

(ii) For each $0 \leq s < t$, the r.v. $B_t - B_s \sim \mathcal{N}(0, t-s)$

(iii) B has independent increments, that is, for any sequence $0 \leq t_1 < t_2 < ... < t_n$, the random variables

$$B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$$

are independent

(iv) Sample paths are continuous a.s.

Obs: If $\mathbb{F}$ contains more information than $\mathbb{F}^B$, (iii) is modified to $B(t) - B(s)$ is independent of $\mathcal{F}_s$ for all $0 < s < t$.

Intuition: Random Walk Representation of a B.M.

- Divide time into discrete periods of length Δ: $\{\Delta, 2\Delta, 3\Delta, ...\}$

- Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables s.t.

$$\mathbb{P}(X_j = h) = \mathbb{P}(X_j = -h) = 1/2$$

- Let $Y^{\Delta,h}(n\Delta) = X_1 + X_2 + ... + X_n$, $Y^{\Delta,h}(0) = 0$: **accumulated sum**

- Interpolate: for $t \in (n\Delta, (n+1)\Delta)$

$$Y^{\Delta,h}(t) = Y^{\Delta,h}(n\Delta)\frac{(n+1)\Delta - t}{\Delta} + Y^{\Delta,h}((n+1)\Delta)\frac{t - n\Delta}{\Delta}$$

- For each (Δ, h) we have a well-defined stochastic process. Let's take $h = \sqrt{\Delta}$ so:
  - For large Δ, jump of $Y^{\Delta,\sqrt{\Delta}} << $ time interval
  - For small Δ, jump of $Y^{\Delta,\sqrt{\Delta}} >> $ time interval

Random Walk Representation of a B.M.

- For the process $(Y^{\Delta,\sqrt{\Delta}}(t))_{t\geq 0}$:

1) Is $Y^{\Delta,\sqrt{\Delta}} = 0$?
2) Does it have independent increments?
3) Does it satisfy $Y^{\Delta,\sqrt{\Delta}}(t) - Y^{\Delta,\sqrt{\Delta}}(s) \sim \mathcal{N}(0, t-s)$?
4) Is it continuous?

Random Walk Representation of a B.M.

- For the process $(Y^{\Delta,\sqrt{\Delta}}(t))_{t\geq 0}$:
  1) Is $Y^{\Delta,\sqrt{\Delta}} = 0$? Yes
  2) Does it have independent increments? Yes
  3) Does it satisfy $Y^{\Delta,\sqrt{\Delta}}(t) - Y^{\Delta,\sqrt{\Delta}}(s) \sim \mathcal{N}(0, t-s)$? No!
  4) Is it continuous? Yes
- Consider the process $Z_t := \lim_{\Delta\to 0} Y^{\Delta,\sqrt{\Delta}}(t)$. Obviously 1, 2 and 4 will be preserved. What about 3?
- Recall: The **characteristic function** $\Phi_X : \mathbb{R} \to \mathbb{R}$ of a random variable X is defined as

$$\Phi_X(\lambda) = \mathbb{E}[e^{i\lambda X}],\ \lambda \in \mathbb{R}$$

- **Fact:** If X and Y are random variables and $\Phi_X(\cdot) = \Phi_Y(\cdot)$, then $X = Y$

Random Walk Representation of a B.M.

- Take $X = Y^{\Delta,\sqrt{\Delta}}(t)$. For $t = n\Delta$

$$\begin{aligned} \Phi_{Y^{\Delta,\sqrt{\Delta}}(t)}(\lambda) &= \prod_{j=1}^{n} \mathbb{E}[e^{i\lambda X_j}] = (\mathbb{E}[e^{i\lambda X_1}])^n \\ &= \left(e^{i\lambda\sqrt{\Delta}}/2 + e^{-i\lambda\sqrt{\Delta}}/2\right)^n \\ &= [\cos(\lambda\sqrt{\Delta})]^n = [\cos(\lambda\sqrt{\Delta})]^{t/\Delta} \end{aligned} \quad (1)$$

- Taking logs and applying L'Hopital rule twice: $\lim_{\Delta\to 0} \frac{t}{\Delta} \log(\cos(\lambda\sqrt{\Delta})) = -t\lambda^2/2$, so

$$\underbrace{\lim_{\Delta\to 0} \Phi_{Y^{\Delta,\sqrt{\Delta}}(t)}}_{=\Phi_{\lim\limits_{\Delta\to 0} Y^{\Delta,\sqrt{\Delta}}(t)} = \Phi_{Z_t}} = e^{\frac{-\lambda^2}{2}t} = \Phi[\mathcal{N}(0,t)](\lambda)$$

- Hence, $Z_t \sim \mathcal{N}(0,t)$, i.e. $Z := (Z_t)_{t\geq 0}$ is a B.M.!
- Differentiability of B.M.? Slope of $Y^{\Delta,\sqrt{\Delta}}(t) = \sqrt{\Delta}/\Delta \to \infty$: nowhere differentiable!

Some Properties of the B.M.

- Gaussian process: $B_t \sim \mathcal{N}(0, t)$ and cov$(B_t, B_s) = \min(t, s)$
- It is a $\mathbb{F}^B$-martingale, i.e., $\mathbb{E}[B_t|\mathcal{F}_s^B] = B_s \ t > s$, where $\mathbb{F}^B$ is the Brownian filtration
- It has independent increments
- Increments are stationary: $(B_{t+t_0} - B_{t_0})_{t \geq 0}$ is also a B.M. for all $t_0 \geq 0$
- It is Markovian: $\mathbb{E}[\cdot|\mathcal{F}_t^B] = \mathbb{E}[\cdot|B_t]$, $t \geq 0$
- $\lim_{t\to\infty} \sup \frac{B_t}{\sqrt{2t \log(\log(t))}} = 1$ and $\lim_{t\to\infty} \inf \frac{B_t}{\sqrt{2t \log(\log(t))}} = -1$
- Almost every path is continuous and nowhere differentiable

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Quadratic Variation

Definition

The quadratic variation of X, denoted $[X]$, is defined as

$$[X]_t := \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2,\ t \geq 0$$

for all partitions $t_0 = 0 < t_1 < ... < t_n = t$ $(\|\Delta_n\| := \sup |t_i - t_{i-1}|)$

Observations:

- If X stochastic process, various forms of the "limit" in the literature. For us $\lim_{\|\Delta_n\|\to 0} \mathbb{E}\left[\left(\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})^2\right)^2\right]$
- $[X]$ could be stochastic or deterministic. Can be computed from data!
- It measures the degree of variability of a stochastic process/function
- $f \in C^1$, $[f] \equiv 0$. B.M: $[B]_t = t$ (wild paths $\Leftrightarrow$ nowhere differentiable).

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