Lecture 2: Stochastic Integration

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How to define Stochastic Integrals?

- Example: $X := (X_t)_{t\in\mathbb{T}}$ stochastic process that governs a stock price
- $(\beta_t)_{t\geq 0}$: trading strategy. Suppose β is step. Trading gains =

$$\beta_0(X_{t_1} - X_{t_0}) + \beta_2(X_{t_2} - X_{t_1}) + \beta_3(X_{t_3} - X_{t_2}) \sim \text{``} \int_0^{t_3} \beta_s dX_s \text{''}$$

- How to define a **stochastic integral**? One option: for each ω $t \mapsto X_t(\omega)$ is a standard function. Let's try that
- Recall: f is Riemann-integrable in $[a, b]$ if the following limit exists:

$$\int_a^b f(t)dt = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(\tau_i)(t_i - t_{i-1})$$

  where $\Delta_n = \{a = t_0, t_1, ..., t_n = b\}$ is a partition of $[a, b]$, $\|\Delta_n\| = \max_{i=1,\ldots,n}(t_i - t_{i-1})$ and τ_i is any point in $[t_{i-1}, t_i]$.
- Recall that if f continuous, then it is Riemann integrable

Riemann-Stieltjes Integral

- Let g be increasing in $[a, b]$;
- A bounded function f in $[a, b]$ is Riemann-Stieltjes (R-S) integrable w.r.t. g if the following limit exists:

$$\int_a^b f(t)dg(t) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^{n} f(\tau_i)(g(t_i) - g(t_{i-1})).$$

- **Fact**: Continuous functions are R-S integrable w.r.t. to any increasing function.
- How to define $\int_a^b g(t)df(t)$ with g increasing and f continuous? Integration by parts:

$$\int_a^b g(t)df(t) := f(t)g(t)\Big|_a^b - \underbrace{\int_a^b f(t)dg(t)}_{\text{well-defined}}$$

- But I want an integrand g more general than just "increasing". **Question**: Can we define R-S integral for any continuous f and g?

Example $f = g$

Left sum $L_n := \sum_{i=1}^{n} f(t_{i-1})(f(t_i) - f(t_{i-1}))$ and right sum $R_n := \sum_{i=1}^{n} f(t_i)(f(t_i) - f(t_{i-1}))$; It is easy to see that

$$\begin{aligned} R_n &= \frac{1}{2}\left(f(b)^2 - f(a)^2 + \sum_{i=1}^{n}(f(t_i) - f(t_{i-1}))^2\right) \\ L_n &= \frac{1}{2}\left(f(b)^2 - f(a)^2 - \sum_{i=1}^{n}(f(t_i) - f(t_{i-1}))^2\right) \end{aligned} \tag{1}$$

And the R-S integral is well-defined if and only if

$$\underbrace{\lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n}(f(t_i) - f(t_{i-1}))^2}_{[f]_t :=} = 0.$$

That is, if and only if the quadratic variation of f is zero!

Example: Continued

- Suppose f is C^1. Then, $[f]_t = 0$
- Suppose f is continuous satisfying $|f(t) - f(s)| \approx |t - s|^{1/2}$
- In this case

$$0 \le R_n - L_n \approx \sum_{i=1}^{n} (t_i - t_{i-1}) = b - a$$

In this case $\int_a^b f(t) df(t)$ is not well-defined.

- Are there any functions like this? Recall: random walk approximation of a B.M.

$$|B_{n\Delta} - B_{(n-1)\Delta}| \approx \sqrt{\Delta}$$

More precisely, since the quadratic variation of a B.M. is strictly larger than zero ($[B]_t = t$), $\int_0^t B_s dB_s$ is not well-defined using the path-by-path (i.e. ω-by-ω) notion

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Martingale Requirement of Stochastic Integral

- Suppose we face a **fair gamble** (a martingale) represented by our BM

- Let $0 = t_0 < t_1 < ... < t_n$ represent times at which we are allowed to change our position

- Suppose we fix a position β_i deterministic at time t_i until t_{i+1}, $i = 1, ..., n$. Accumulated gains process:

$$V_{t_j} = V_0 + \sum_{i=0}^{j} \beta_i (B_{t_{i+1}} - B_{t_i}),\ j = 0, ..., n,$$

  is also a martingale

- Any sensible economic model of monetary gains should have this feature (fair gamble $\Rightarrow$ gains follow a martingale). Our notion of stochastic integral must have the martingale property

Wiener Integral

- Notion of stochastic integral for **deterministic integrands** β
- Recall $\mathbb{T} = \mathbb{R}_+$ or $[0, T]$, $T > 0$
- A step function $f : \mathbb{T} \to \mathbb{R}$ is of the form

$$f(t) = \sum_{j=0}^{n} a_j \mathbb{1}_{]t_j, t_{j+1}]}(t),$$

  where $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < ... < t_n < \infty$, $a_j \in \mathbb{R}$

- For a step function f define the r.v.

$$I(f) := \sum_{j=0}^{n-1} a_j (B_{t_{j+1}} - B_{t_j})$$

- Observe that $I(f)$ is a normally-distributed random variable. Furthermore, $\mathbb{E}[I(f)^2] = \int_0^\infty f^2(t)dt < \infty$, so $I(f)$ has finite second moment ($\in L^2(\Omega, \mathcal{F}, \mathbb{P})$ the space of square integrable r.v.);

Extension to $L^2(\mathbb{T})$

- Set of square-integrable functions over $\mathbb{T}$:

$$L^2(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{R} | \int_{\mathbb{T}} f^2(t)dt < \infty\}$$

- **Fact 1:** Step functions are dense in $L^2(\mathbb{T})$: for $f \in L^2(\mathbb{T})$ there is $(f_n)_{n\in\mathbb{N}}$ sequence of step functions converging to f.

- Compute $I(f_n)$ for f_n (we can do that). Recall that $\mathbb{E}[I(f_n)^2] < \infty$ (i.e. $I(f_n) \in L^2(\Omega)$, the set of r.v. with finite second moment)

- **Fact 2:** $\lim_{n\to\infty} I(f_n)$ exists in the following sense: there exists a random variable Z such that
  - $\mathbb{E}[Z^2] < \infty$ and $\mathbb{E}[(I(f_n) - Z)^2] \to 0$ as $n \to \infty$
  - For $g_n \neq f_n$ also converging to f, $\mathbb{E}[(I(g_n) - Z)^2] \to 0$ as well.

- The limit Z depends on f, so we denote it $I(f)$

Properties

Theorem

$I : L^2(\mathbb{T}, \mathcal{B}(\mathbb{T})) \to L^2(\Omega)$ is linear. Moreover, for any $f \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ (i.e. square-integrable)

$$I(f) \sim \mathcal{N}\left(0, \int_{\mathbb{T}} f^2(t)\,dt\right)$$

Corollary

$f, g \in L^2(\mathbb{T})$. Then cov$(I(f), I(g)) = \int_{\mathbb{T}} f(t)g(t)\,dt :=< f, g >$. Since the Wiener integral is Gaussian, $< f, g >= 0$ implies that $I(f)$ and $I(g)$ are independent.

Wiener integral

So far: stochastic integral defined over the whole set of times $\mathbb{T}$.

Definition

For $a < b \in \mathbb{T}$, the **Wiener integral** is defined as

$$\int_a^b f(t)dB_t := I(f \cdot \mathbb{1}_{]a,b]})$$

Theorem

For $f \in L^2(\mathbb{T})$, the process

$$M_t := \int_0^t f(u)dB_u, \ t \geq 0$$

is a Gaussian process with zero mean and variance $\int_0^t f(u)^2 du$. Furthermore, it has independent increments and it is a martingale.

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Ito Integral

Idea: As the Wiener integral but integrand (e.g. strategy or assets holding) is stochastic. However, not any integrand will work. Example: trading strategy needs to be adapted to current information.

Definition

A process $B := (B_t)_{t\geq 0}$ is on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is an $\mathbb{F}$-Brownian motion if it satisfies (i), (ii) and (iv) in the old definition plus $B_t - B_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$.

(allow the agent to have potentially more info than $\mathbb{F}^B$)

Construction of Ito Integral

- **Step process**: $f_t(\omega) = \sum_{j=0}^{n-1} a_j(\omega)\mathbb{1}_{]t_j,t_{j+1}]}(t)$, with $a_j \in L^2(\Omega, \mathcal{F}_{t_j})$, i.e. strategy a_j over $]t_j, t_{j+1}]$ is square integrable and adapted to $\mathcal{F}_{t_j}$ (left end-point of the time interval!)
- For step process define $I(f) := \sum_{j=0}^{n-1} a_j(B(t_{j+1}) - B(t_j))$. It's a r.v.
- Verify that for f step process

$$\mathbb{E}[I(f)^2] = \mathbb{E}\left[\int_{\mathbb{T}} f_t^2(\omega)\,dt\right] = \mathbb{E}\left[\sum_{j=0}^{n-1} a_j^2(t_{j+1} - t_j)\right] < \infty$$

  so $I(f)$ is a square-integrable r.v. (i.e. $I(f) \in L^2(\Omega)$)
- Let $\mathcal{H}^2 = \left\{(f_t)_{t\geq 0} \middle|\, \text{f is } \mathbb{F}-\text{adapted and } \mathbb{E}\left[\int_{\mathbb{T}} f_t^2(\omega)\,dt\right] < \infty\right\}$

Ito Integral

- **Fact 1:** Step processes are dense in $\mathcal{H}^2$
- For $f \in \mathcal{H}^2$, take $(f_n)_{n\in\mathbb{N}}$ converging to f. **Fact 2:** The limit $\lim_{n\to\infty} I(f_n)$ (in $L^2(\Omega)$ norm) exists and does not depend on the approximating sequence (i.e., **well-defined**). Call it $I(f)$ (it is a r.v.!)

Theorem

For all $f \in \mathcal{H}^2$: $\mathbb{E}[I(f)] = 0$ and $\mathbb{E}[I(f)^2] = \mathbb{E}\left[\int_{\mathbb{T}} f_t^2(\omega)\,dt\right] < \infty$.

Definition

The **Ito integral** is defined as follows: for $f \in \mathcal{H}^2$, $0 \le a < b < \infty$

$$\int_a^b f_t\,dB_t := I(f \cdot \mathbb{1}_{]a,b]})$$

Finally...

Theorem

Let f be a step process. The process

$$M_t := \int_0^t f_s dB_s,\ t \geq 0$$

is a continuous $\mathbb{F}-$martingale. If $f \in \mathcal{H}^2$, then M_t is an $\mathbb{F}-$martingale that can be chosen to be continuous. Its mean is zero. Variance is $\mathbb{E}\left[\int_0^t f_s^2 ds\right] < \infty$ for all $t \geq 0$.

Observation: Where we evaluate the partial sums of the stochastic integral matters a lot. Accumulated gains will be a martingale if and only if step functions are constructed using weights (i.e., r.v. a_i) that are adapted to the left-hand side of the time intervals. Economically this makes a lot of sense.

Example

- $\int_0^T B_t dB_t$? Let's use the Ito integral
- Notation: $L^2 \lim X_n := \lim_{n\to\infty} \mathbb{E}[X_n^2]$

$$
\begin{aligned}
\int_0^T B_t dB_t &= L^2 \lim \sum_{j=0}^{n-1} B_{\frac{j}{n}T}(B_{\frac{j+1}{n}T} - B_{\frac{j}{n}T}) \\
&= L^2 \lim \frac{1}{2} \sum_{j=0}^{n-1} (B^2_{\frac{j+1}{n}T} - B^2_{\frac{j}{n}T}) \\
&\quad - \underbrace{\frac{1}{2} \sum_{j=0}^{n-1} \left(B_{\frac{j+1}{n}T} - B_{\frac{j}{n}T}\right)^2}_{L^2 \lim = [B]_T = T} \\
&= \frac{1}{2} B_T^2 - \underbrace{\frac{1}{2} T}_{\text{new term!}}
\end{aligned} \tag{2}
$$

- Check that it is a martingale.

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Ito Processes

Definition

An **Ito process** is a stochastic process of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dB_s,\ \ t \in \mathbb{T}$$

s.t. $b \in \mathcal{H}^2$ (finite variance of X_t for all t) and $a \in \mathcal{H}^1$, where $\mathcal{H}^1$ is the set of adapted processes s.t.

$$\mathbb{E}\left[\int_{\mathbb{T}} |a_s| ds\right] < \infty$$

(ensures finite first moment).

Short notation: $dX_t = a_t dt + b_t dB_t$.

Ito Formula: Chain Rule

- **Chain rule in basic calculus**: Suppose $f, g : \mathbb{R} \to \mathbb{R}$ deterministic and differentiable. Then

$$d[f(g_t)] = f'(g_t) g'_t dt, \text{ or, } df(g_t) = f'(g_t) dg_t$$

$\Rightarrow$ all other derivatives of f are accompanied by term of order $o(dt^2)$. This is because g is differentiable so it does not vary that much as time evolves (technically speaking: quadratic variation $[g]_t = 0$)

- B.M. has a wild behavior, so higher derivatives of f can be accompanied by first order terms $o(dt)$

- Recall that $B_t^2 = 2\int_0^t B_s dB_s + t$. So for B_t playing the role of g and $f(x) = x^2$

$$\underbrace{d(B_t^2)}_{\text{“}df(B_t)\text{”}} = \underbrace{2B_t dB_t}_{\text{“}f'(B_t)dB_t\text{”}} + \underbrace{dt}_{\text{new term } d[B]_t}$$

Ito Formula: First Version

Theorem

$f \in C^2(\mathbb{R})$. Then

$$f(B_t) = f(B_0) + \int_0^t f'(B_u)\,dB_u + \underbrace{\frac{1}{2}\int_0^t f''(B_u)\,du}_{\textit{new term}}$$

Notation: $df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$. $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$. Then,

$$\begin{aligned} f(t, B_t) &= f(0, B_0) + \int_0^t \frac{\partial f}{\partial x}(u, B_u)\,dB_u + \int_0^t \frac{\partial f}{\partial t}(u, B_u)\,du \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(u, B_u)\,dt \end{aligned} \tag{3}$$

Notation: $df(t, B_t) = \partial_x f(t, B_t)dB_t + \partial_t f(t, B_t)dt + \frac{1}{2}\partial_{xx} f(t, B_t)dt$

Obs: In both cases, $f(B_t)$ and $f(t, B_t)$ are Ito process

Ito Formula for Ito Processes

Theorem

$f \in C^2(\mathbb{R})$ and $dX_t = a_t dt + b_t dB_t$ Ito. Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_u)\,dX_u + \frac{1}{2}\int_0^t f''(X_u)\,d[X]_t$$

where $[X]_t := \int_0^t b_u^2\,du$ is the quadratic variation of X_t, $t \in \mathbb{T}$. Hence,

$$\int_0^t f''(X_u)\,d[X]_t = \int_0^t f''(X_u) b_u^2\,du.$$

Thus, $Y_t = f(X_t)$ is an Ito process. Notation:

$$\begin{aligned} dY_t = df(X_t) &= f'(X_t)\,dX_t + \frac{1}{2} b_t^2 f''(X_t)\,dt \\ &= \left[f'(X_t) a_t + \frac{1}{2} b_t^2 f''(X_t)\right] dt + f'(X_t) b_t\,dB_t \end{aligned}$$

Multidimensional Formula

Theorem

Let $dX_t = a_t dt + b_t dB_t$ *be an Itô process in* $\mathbb{R}^n$ *where* $a_t \in \mathbb{R}^n$*,* $b_t \in \mathbb{R}^{n \times d}$ *and* B_t *a d-dimensional Brownian motion. Let* $g : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^p$ *of class* $C^{1,2}$*. The process* $Y_t = g(X_t)$ *satisfies*

$$dY_t^k = \frac{\partial g^k}{\partial t} dt + \sum_i \frac{\partial g^k}{\partial x^i} dX_t^i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 g^k}{\partial x^i \partial x^j} < dX^i, dX^j >_t \qquad (4)$$

where $< X^i, X^j >_t = \sum_{l=1}^{d} \int_0^t b^{i,l} b^{j,l} dt$ *is the cross-variation between* X^i *and* X^j*.*

Connection Between Martingales and Ito Processes

Let X_t be an Ito process of the form $dX_t = \mu_t dt + \sigma_t dB_t$ with $\sigma \in \mathcal{H}^2$.

- If $\mu_t = 0$ a.s. for all t, then X is a martingale;
- If $\mu \geq 0$ a.s. for all t, then X is a sub-martingale;
- If $\mu \leq 0$ a.s. for all t, then X is a super-martingale;

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