# Lecture 3: Connection with Differential Equations

## Stochastic Differential Equations and Feynman-Kac Formula

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# Motivation I: Discounted Present Values

- A firm's cash flow at time $t$ is given by $f(X_t)$, where $X := (X_t)_{t \geq 0}$ is a (random) **payoff-relevant state variable**. E.g.: the price of oil
- What is the firm's current value if today's price of oil is $x > 0$?

$$\underbrace{F(x)}_{=:\text{Firm's value}} := \mathbb{E}\left[\int_0^\infty e^{-rt} f(X_t)\, dt \,\Big|\, X_0 = x\right] \tag{1}$$

- Calculating this value directly is hard:
  - Requires knowing the distribution of $Y_t := f(X_t)$ for all $t \geq 0$
  - Calculating the NPV path by path, and then take the average
  - Finally, do this for all $x > 0$!
- Can we find an indirect method? A recursive expression for $F(x)$, i.e., a differential equation is much simpler

# Motivation II: Stochastic Differential Equations (SDE)

- Suppose $dX_t = \mu_t dt + \sigma_t dB_t$ is an Ito process taking values in $A \subseteq \mathbb{R}$, with $\mu$ and $\sigma$ stochastic processes
- Simpler case: If $\mu, \sigma : A \to \mathbb{R}$ (i.e. they are standard functions!) with

$$\mu_t = \mu(X_t) \quad \text{and} \quad \sigma_t = \sigma(X_t) \tag{2}$$

  then we say that $X := (X_t)_{t \geq 0}$ satisfies a **SDE**.
- If $X$ satisfies a SDE, it is likely that $F(x)$ and $F(x + \Delta)$ ($\Delta$) small, are also connected via an equation
- The previous discussion motivates the study of stochastic differential equations and of the equations associated to the solutions to SDEs

# Back to Firm Valuation

- Suppose that the price of oil evolves according to the SDE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t,\ t > 0,\ X_0 = x.$$

- Notice that

$$F(x_t) = f(X_t)\,dt + \underbrace{e^{-rdt}}_{\approx 1 - rdt}\,\mathbb{E}[F(X_{t+dt})]$$

- By Ito's rule, for $dt$ small:

$$\underbrace{dF(X_t)}_{F(X_{t+dt})-F(X_t)\approx} \qquad = F'(X_t)\,dX_t + \frac{1}{2}[\sigma(X_t)]^2 F''(X_t)\,dt$$

$$\Rightarrow \qquad \mathbb{E}[F(X_{t+dt})|\mathcal{F}_t] = F(X_t) + [F'(X_t)\mu(X_t) + \frac{1}{2}[\sigma(X_t)]^2 F''(X_t)]\,dt$$

where we used that $\mathbb{E}[F'(X_t)\sigma(X_t)\,dB_t|\mathcal{F}_t] = 0$ (property of Ito integral!)

# An Equation for $F(\cdot)$

- Hence, back in $F(x_t) = f(X_t)dt + (1 - rdt)\mathbb{E}[F(X_{t+dt})]$:

$$\begin{aligned} \cancel{F(X_t)} &= f(X_t)dt + \cancel{F(X_t)} + [F'(X_t)\mu(X_t) + \frac{1}{2}[\sigma(X_t)]^2 F''(X_t)]dt \\ &\quad - rF(X_t)dt - r\left[F'(X_t)\mu(X_t) + \frac{1}{2}[\sigma(X_t)]^2 F''(X_t)\right](dt)^2 \end{aligned}$$

- So dividing by $dt$ and making $dt \to 0$ we obtain:

  $rF(X_t) = f(X_t) + F'(X_t)\mu(X_t) + \frac{1}{2}[\sigma(X_t)]^2 F''(X_t)$, for all $X_t$

- Thus, we obtain the following ordinary differential for F:

$$rF(x) = f(x) + F'(x)\mu(x) + \frac{1}{2}[\sigma(x)]^2 F''(x)$$

  Finance: **arbitrage** interpretation of this equation

  - $rF(x)$ : interest earned if I sell the firm/asset
  - $f(x)$ : dividend I receive if I hold the firm/asset
  - $\mathbb{E}[dF(X_t)] = F'(x)\mu(x) + \frac{1}{2}[\sigma(x)]^2 F''(x)$ : Expected capital gains if hold the firm/asset

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# SDE

- A SDE taking values in $\mathbb{R}^n$ is an Ito process of the form:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \ t \geq 0$$

  where:

  - $B = (B_1, ..., B_d)$ a $d-$dimensional B.M. (i.e. a vector of independent B.M.s)
  - Vector process

$$b(t, x, \omega) = (b^1(t, x, \omega), ...b^n(t, x, \omega))_{t \in \mathbb{T}} : \mathbb{T} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n;$$

  - Matrix process ($i$: rows, $j$: columns)

$$\sigma(t, x, \omega) = (\sigma^{ij}(t, x, \omega))_{t \in \mathbb{T}} : \mathbb{T} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^{n \times d};$$

- Or component-wise:

$$dX_t^i = b^i(t, X_t)dt + \sum_{j=1}^{d} \sigma^{ij}(t, X_t)dB_t^j, \ i = 1, ..., n;$$

# Diffusions and Controlled Diffusions

- Recall $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,\ t \geq 0$
  - If $b$ and $\sigma$ are deterministic functions (i.e., $b, \sigma : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$), we call the solution $X := (X_t)_{t\geq 0}$ to the SDE a **diffusion**
  - If random, we assume $b = \tilde{b}(t, x, \alpha_t(\omega))$ and $\sigma = \tilde{\sigma}(t, x, \alpha_t(\omega))$ where $\tilde{b}, \tilde{\sigma}$ are deterministic and $\alpha := (\alpha_t)_{t\in\mathbb{T}}$ is (a bit more than) adapted to the corresponding filtration. We say that the SDE is a **controlled diffusion** by $\alpha$.
- A **solution** to this SDE is any process with finite first and second moment such that

$$X_s = X_t + \int_t^s b(u, X_u)du + \int_t^s \sigma(u, X_u)dB_u,\ t \leq s \in \mathbb{T}$$

- In general, under mild conditions on $b$ and $\sigma$, solutions to SDEs exist!

▸ SDEs: Formal Analysis and Existence

# Examples of SDEs: Deterministic Coefficients

- Geometric Brownian motion: $\mu \in \mathbb{R}$ and $\sigma > 0$,

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

- Mean-reverting process: $\kappa > 0$, $\sigma > 0$:

$$dX_t = -\kappa(X_t - \overline{x})dt + \sigma dB_t$$

- CIR model (interest rates):

$$dX_t = -\kappa(X_t - \overline{x})dt + \sigma\sqrt{X_t}dB_t$$

$\overline{x}$ is a long-run mean trend.

**Exercise:** Find solutions for the geometric and mean-reverting cases

# Examples of SDEs: Random Coefficients

- A worker that is able to produce output $X$ by exerting effort $a$:

$$dX_t = \mu(X_t, a_t)\,dt + \sigma\,dB_t;$$

- The agent observes past output realizations, and hence her effort decision today may be influenced by his/her past performance
- If this is the case (i.e. $a_t$ is a function of the path $(X_s : \ s \leq t)$), his effort at any time is random $\big(a_t = a_t(\omega)\big)$. Hence, the strategy $a := (a_t)_{t\geq 0}$ is a stochastic process
- This is very important from the perspective of economics: we make **dynamic decisions** (e.g. effort) that affect **payoff-relevant state variables** (output, which determines wages) that can be modeled via **stochastic processes**

# Markov Property of Diffusions

## Theorem

*Let $X := (X_t)_{t\geq 0}$ be a diffusion taking values in $\mathbb{R}^d$. Then,*

- *The solution to the SDE is adapted to $\mathbb{F}^B$*
- *The solution satisfies the **Markov property**, that is, for any (borelian) bounded function $g$ on $\mathbb{R}^d$ and $s \geq t$,*

$$\mathbb{E}[g(X_s)|\mathcal{F}_t] = \mathbb{E}[g(X_s)|X_t]$$

**Example:** The value of a firm at time $t$ is, by definition, given by

$$\mathbb{E}\left[\int_t^\infty e^{-r(s-t)} f(X_s)\,ds \Big| \mathcal{F}_t\right] = \int_t^\infty e^{-r(s-t)} \mathbb{E}[f(X_s)|\mathcal{F}_t]\,ds$$

$$\underbrace{=}_{\text{Markov}} \int_t^\infty e^{-r(s-t)} \mathbb{E}[f(X_s)|X_t]\,ds = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} f(X_s)\,ds \Big| X_t\right]$$

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# Feynman-Kac: Motivation

- Recall that $F(x) = \mathbb{E}\left[\int_0^\infty e^{-rt} h(X_t)\, dt \,\middle|\, X_0 = x\right]$ satisfies the ODE

$$0 = -rF(x) + h(x) + F'(x)\mu(X) + \frac{1}{2}\sigma(X)^2 F''(x)$$

  when $dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t$ in $\mathbb{R}$. Nice, because
  - It connects diffusions with ODEs (or, more generally, PDEs)
  - We can apply numerical methods to solve the above ODE

- **Natural question:** Given the above ODE, is

$$\mathbb{E}\left[\int_0^\infty e^{-rt} h(X_t)\, dt \,\middle|\, X_0 = x\right]$$

  the unique solution to this ODE?

- **Feynman-Kac Theorem says that in general, the answer is yes**
  - The theorem provides a probabilistic representation to solutions to differential equations

# Feynman-Kac Formula: Infinite Horizon

Let $\mu, \sigma, h : \mathbb{R} \to \mathbb{R}$ and $r > 0$. Consider the **ordinary differential equation (ODE)**

$$
\begin{aligned}
F'(x)\mu(x) + \frac{1}{2}\sigma(x)^2 F''(x) - rF(x) + h(x) &= 0, \ x \in \mathbb{R} \qquad (3) \\
\lim_{t\to\infty} \mathbb{E}[e^{-rt} F(X_t)] &= 0, \qquad (4)
\end{aligned}
$$

Theorem

*(**Feynman-Kac Formula**) Suppose that $F \in C^2(\mathbb{R})$ solves (3)-(4) and that satisfies a growth condition (i.e., $|F(x)| \leq C(1 + x^2)$ for some $C > 0$). Then*

$$F(x) = \mathbb{E}\left[\int_0^\infty e^{-rt} h(X_t) dt | \ X_0 = x\right]$$

*where $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$.*

▸ Feynman-Kac for Multidimensional SDE and Finite/Infinite Horizon: Connection with PDEs

# Sketch of the Proof

Define $W_t := \int_0^t e^{-rs} h(X_s) ds + e^{-rt} F(X_t)$ with $F$ a solution to the ODE (3)-(4). Applying Ito's rule to $W$ we obtain that

$$dW_t = e^{-rt} \underbrace{\left[h(X_t)dt - rF(X_t) + F'(X_t)\mu(X_t) + \frac{1}{2}\sigma(x)^2 F''(x)\right.}_{=0,\text{ as f solves the PDE}}$$
$$+ e^{-rt} F'(X_t)\sigma dB_t.$$

Hence, $W := (W_t)_{t \geq 0}$ is a martingale. Consequently

$$\mathbb{E}[W_t] = W_0 = F(X_0), \ \forall t \geq 0.$$

But since $\lim_{t \to \infty} \mathbb{E}[e^{-rt} F(X_t)] = 0$, then:

$$F(X_0) = \mathbb{E}[W_t] = \mathbb{E}\left[\int_0^t e^{-rs} h(X_s) ds + e^{-rt} F(X_t)\right] \to \mathbb{E}\left[\int_0^\infty e^{-rs} h(X_s) ds\right]$$

as $t \to \infty$, from where we conclude.

# Example I: Arithmetic Brownian Motion

- $dX_t = \mu dt + \sigma dB_t$, with $\mu, \sigma \in \mathbb{R}$ (i.e., $X_t \in \mathbb{R}$)
- Associated ODE is linear:

$$0 = -rF(x) + h(x) + F'(x)\mu + \frac{1}{2}\sigma^2 F''(x)$$

  Solution: Homogeneous + particular

- Homogeneous solution (i.e., $h \equiv 0$): $F_1(x) = Ae^{-\alpha x} + Be^{\beta x}$ where $-\alpha$ and $\beta$ solve $r - \mu\xi - \frac{1}{2}\sigma^2\xi^2 = 0$
- Particular? Suppose $h(x) = \exp(\lambda x)$. Try $F_2(x) = K\exp(\lambda x)$. Plugging this in the ODE:

$$K = \frac{1}{r - \lambda\mu - \frac{1}{2}\sigma^2\lambda^2}$$

- **Q**: How do we determine $A$ and $B$? How large can $\lambda > 0$ be?

# Example II: Geometric Brownian Motion

- $dX_t = \mu X_t dt + \sigma X_t dB_t$, $X_0 > 0$, with $\mu, \sigma \in \mathbb{R}$ (i.e., $X_t \in \mathbb{R}$)
- Associated ODE is linear:

$$0 = -rF(x) + h(x) + F'(x)x\mu + \frac{1}{2}x^2\sigma^2 F''(x)$$

  Solution: Homogeneous + particular

- Homogeneous solution (i.e., $h \equiv 0$): $F_1(x) = Cx^{-\gamma} + Dx^{\delta}$ where $-\alpha$ and $\beta$ solve $r - \mu\xi - \frac{1}{2}\sigma^2\xi(\xi - 1) = 0$
- Particular? Suppose $h(x) = x^{\lambda}$, $\lambda > 0$. Try $F_2(x) = Kx^{\lambda}$. Plugging this in the ODE:

$$K = \frac{1}{r - \lambda\mu - \frac{1}{2}\sigma^2\lambda(\lambda - 1)}$$

- **Q**: How do we determine $C$ and $D$? How large can $\lambda > 0$ be?

# Bibliography


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# Formal Analysis of SDEs: Preliminaries

- $\mathbb{T} = [0, T]$ or $[0, \infty)$
- Filtered probability space: $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$
- $B = (B_1, ..., B_d)$ a $d-$dimensional B.M. w.r.t. $\mathbb{F}$
- Vector process

$$b(t, x, \omega) = (b^1(t, x, \omega), ...b^n(t, x, \omega))_{t \in \mathbb{T}} : \mathbb{T} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$$

- Matrix process ($i$: rows, $j$: columns)

$$\sigma(t, x, \omega) = (\sigma^{ij}(t, x, \omega))_{t \in \mathbb{T}} : \mathbb{T} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^{n \times d}$$

- For all $x$, $b(\cdot, x, \cdot)$ and $\sigma(\cdot, x, \cdot)$ are progressively measurable
- For $\omega$, $b(\cdot, \cdot, \omega)$ and $\sigma(\cdot, \cdot, \omega)$ are borelian (this just means that I can integrate them in $\mathbb{T} \times \mathbb{R}^d$)

# Strong Solution Concept

Let

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \ t \geq 0 \tag{5}$$

## Definition

A strong solution of the SDE (5) starting at time $t$ is a vectorial progressively measurable process $X = (X^1, ..., X^d)$ such that

$$\int_t^s |b(u, X_u)|du + \int_t^s |\sigma(u, X_u)|^2 du < \infty, \ a.s., \ \forall t \leq s \in \mathbb{T}$$

and the relation

$$X_s = X_t + \int_t^s b(u, X_u)du + \int_t^s \sigma(u, X_u)dB_u, \ t \leq s \in \mathbb{T}$$

is true a.s.

# Existence Result

Suppose that $\exists K > 0$ and a process $\kappa$ s.t. for all $t \in \mathbb{T}, \omega \in \Omega,\ x, y \in \mathbb{R}^n$:

$$|b(t,x,\omega) - b(t,y,\omega)| + |\sigma(t,y,\omega) - \sigma(t,y,\omega)| \leq K|x-y|, \tag{6}$$

$$|b(t,x,\omega)| + |\sigma(t,x,\omega)| \leq k_t(\omega) + K|x|, \tag{7}$$

$$\mathbb{E}\left[\int_0^t |\kappa_u|^2 du\right] < \infty,\ \forall t \in \mathbb{T} \tag{8}$$

## Theorem

*If the above hold,* $\exists$ *for all* $t \in \mathbb{T}$ *a strong solution to the SDE (5) starting at* $t$. *Moreover, given any* $\mathcal{F}_t-$*measurable r.v.* $\xi$, $\mathbb{E}[|\xi|^p] < \infty$, $p > 1$, *there is a unique strong solution with initial value* $\xi$ *a.s. Uniqueness is in the indistinguishable sense. The solution satisfies the growth bound* $\mathbb{E}[\sup_{t \leq s \leq T} |X_s|^p] \leq C_T(1 + |\xi|^p)$, *for some* $C_T$, $T \geq t$.

▸ Back to Diffusions

# Feynman-Kac in the General Case: Multimensional SDE and Infinite/Finite Horizon

Connection with PDEs

# Feynman-Kac Formula: Infinite Horizon

Consider the **partial differential equation (PDE)**

$$\begin{aligned} \mathcal{L}F(x) - rF(x) + h(x) &= 0, \; x \in \mathbb{R}^n & (9) \\ \lim_{T \to \infty} \mathbb{E}[e^{-rt}F(X_t)] &= 0, & (10) \end{aligned}$$

where $\mathcal{L}f(x) := V_x(x)\mu(x) + \frac{1}{2}\mathrm{tr}(\sigma(x)\sigma(x)' V_{xx}(x))$

## Theorem

*(**Feynman-Kac Formula**) Suppose that $F \in C^2(\mathbb{R}^n)$ solves (9)-(10) and that satisfies a quadratic growth condition. Then*

$$F(x) = \mathbb{E}\left[\int_0^\infty e^{-rt}h(X_t)dt |\; X_0 = x\right]$$

*where $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$.*

# Finite-Horizon Case: Ingredients

- $\mathbb{T} = [0, T]$
- $B$ is a $d-$dimensional B.M.

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t$$

  where
  - $\mu : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$: drift process (vector)
  - $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$: volatility process (matrix)
- $r : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$; $r(t, X_t)$ : discount or interest rate at $t \geq 0$
- $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$; $h(t, X_t)$: flow payoff at $t$ (e.g.: firm's cash flow or asset dividend)
- $g : \mathbb{R}^d \to \mathbb{R}$; $g(X_T)$ terminal payoff (e.g.: liquidation value of the asset at $T$)
- $\phi_t^\tau = \exp\left(-\int_t^\tau r(s, X_s)\,ds\right)$, $\tau \geq t$: discount factor

# Feynman-Kac Formula: Finite Horizon

Consider the **partial differential equation (PDE)**

$$\mathcal{L}F(t,x) - F(t,x)r(t,x) + h(t,x) = 0,\ (t,x) \in [0,T) \times \mathbb{R}^n \quad (11)$$

$$F(T,x) = g(x),\ x \in \mathbb{R}^n \quad (12)$$

$\mathcal{L}F(t,x) := F_t(t,x) + F_x(t,x)\mu(t,x) + \frac{1}{2}\mathrm{tr}(\sigma(t,x)\sigma(t,x)' F_{xx}(t,x))$ and $\mathrm{tr}(\sigma(t,x)\sigma(t,x)' F_{xx}(t,x)) = \sum_{k=1}^{d} \sigma^{ik}\sigma^{jk}\partial_{x_i x_j} V(t,x)$.

Theorem

***(Feynman-Kac Formula)*** *Under mild growth conditions on $F$ and $g$, if $F \in C^{1,2}([0,T) \times \mathbb{R}^n) \times C^0([0,T] \times \mathbb{R}^n)$ solves (11)-(12) and satisfies a growth condition then*

$$F(t,x) = \mathbb{E}\left[\int_t^T \phi_t^s h(s,X_s)\,ds + \phi_t^T g(x_T) \middle|\, X_t = x\right]$$

# Idea of the Proof

Define $W_t := \int_0^t \phi_0^s h(s, X_s)\,ds + \phi_0^t F(t, X_t)$ with $F$ solving (11)-(12). Applying Ito's rule to $W$ we obtain that

$$dW_t = \phi_0^t \underbrace{[h(t, X_t)\,dt - r(t, X_t)F(t, X_t) + \mathcal{L}f(t, X_t)]}_{=0\text{, as f solves the PDE}} + \phi_0^t V_x \sigma(t, X_t)\,dB_t.$$

Hence, $W := (W_t)_{t \geq 0}$ is a martingale. Now it is easy to see that

$$\mathbb{E}\left[W_T | \mathcal{F}_t\right] = \underbrace{\int_0^t \phi_0^s h(s, X_s)\,ds}_{:= W_t - \phi_0^t f(t, X_t)} + \mathbb{E}\left[\int_t^T \phi_0^s h(s, X_s)\,ds + \phi_0^T g(X_T) \middle| \mathcal{F}_t\right]$$

But since $\mathbb{E}[W_T | \mathcal{F}_t] = W_t$ ($W$ is a martingale)

$$\phi_0^t F(t, X_t) = \mathbb{E}\left[\int_t^T \phi_0^s h(s, X_s)\,ds + \phi_0^T g(X_T) \middle| X_t\right], \text{but } \phi_s^t = \phi_0^s / \phi_0^t.$$

Back to One-Dimensional Feynman-Kac