Lecture 4: Dynamic Programming

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Motivation

- In the last lecture we saw that there is a close connection between

$$F(x) := \mathbb{E}\left[\int_0^\infty e^{-rt} f(X_t)\, dt \middle| X_t = x\right]$$

where $dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t$ and the ODE

$$0 = rw(x) + f(x) + w'(X_t)\mu(X_t) + \frac{1}{2}\sigma(X_t)^2 w''(X_t)$$

Namely, in most cases of interest, F as above will be the unique solution to the previous ODE

- This is great, as we can solve an ODE/PDE for the valuation of a complicated expression
    - ODEs/PDEs have been largely studied
    - Numerical methods for solving them exist

Motivation

- What if we can affect X in order to maximize a expression like F?
- More specifically, can we find a connection between

$$V(x) := \max_{(a_t)_{t \geq 0}} \mathbb{E}\left[\int_0^\infty e^{-rt} f(X_t, a_t)\,dt \middle| X_t = x\right]$$

  where $dX_t = \mu(X_t, a_t)\,dt + \sigma(X_t, a_t)\,dB_t$ and a certain ODE?
- We would be able to solve a complex optimization problem using well-known methods for differential equations
- The theory of **dynamic programming** gives a positive answer to this question

Overview

- We will study **dynamic optimization** problems in which an **agent controls a diffusion**
- Main result: the problem of **finding optimal policies can be reduced to solving differential equations**
  - Second-order ODE for a one-dimensional diffusion in infinite horizon
  - PDE for a multidimensional SDE, or for a finite-horizon problem
- Two advantages:
  - There are standard methods for solving these equations
  - Second, and most importantly, we can more gain intuition and new insights from the cleanness/elegance of the continuous-time framework.

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Example 1: Dynamic Portfolio Choice (Merton, 1971)

- Assume agent's objective is to maximize her final utility of wealth over a period $\mathbb{T} = [0, T]$
- She has access to $n + 1$ assets that evolve as geometric Brownian motions: For $i = 1, ..., n$

$$dX_t^i = \mu^i X_t^i dt + X_t^i \sigma^i dB_t, \ X_0^i > 0$$

  where μ^i is a constant, σ^i is a row vector of size d and B is a d-dimensional Brownian motion. X^0 corresponds to the value of a short-term risk-less bond: $dX_t^0 = rX_t^0 dt$, $r < \mu^i$ for each i
- The agent starts with wealth W_0 and there is no inflow/outflow of funds (i.e., strategies must be *self-financing*)
- How does wealth evolves over time?

Example 1: Continued

- Self-financing condition $\Rightarrow$ The agent chooses a positive vector process $\alpha := (\alpha_t^1, ..., \alpha_t^n)_{t\geq 0}$, s.t. $\overline{\alpha}_t := \sum_{i=1}^{n} \alpha_t^i \leq 1$ for all $t \in [0, T]$ representing the fraction of current wealth W_t invested in each asset (so $1 - \overline{\alpha}_t$ fraction of current wealth on the risky asset);

- Show that the wealth process evolves according to

$$\frac{dW_t}{W_t} = [\alpha \cdot (\mu - r\vec{e}) + r]dt + \alpha \cdot \sigma \, dB_t, \;\; W_0 > 0 \tag{1}$$

  where $\vec{e} = (1, ..., 1)^T$ of size d

- Agent's problem $\max_{\alpha} \mathbb{E}[U(T, W_T)]$ s.t. (1)

- How to solve a problem like this one?

Example 2: Consumption and Infinite Horizon

- In the same context, suppose that, in addition, the also agent consumes part of her wealth every period.
- Show that wealth evolves according to

$$\frac{dW_t}{W_t} = \left(\alpha \cdot (\mu - r\vec{e}) + r - \frac{C_t}{W_t}\right) dt + \alpha \cdot \sigma \, dZ_t, \;\; W_0 > 0 \tag{2}$$

  where $C := (C_t)_{t \geq 0}$ denotes the consumption process.
- Problem: if $\mathbb{T} = [0, \infty)$

$$\max_{(\alpha, C)} \mathbb{E}\left[\int_0^{\infty} e^{-\rho t} u(c_t) dt\right]$$

  s.t. (2). If $\mathbb{T} = [0, T]$, $\max\limits_{(\alpha, C)} \mathbb{E}\left[\int_0^T u(t, c_t) dt + \psi(W_T)\right]$. ρ is the discount rate.

Example 3: Optimal Exercise of American Options

- Suppose we have an American call option of a stock
  - A holder of a call option has the option, but not the obligation, to buy at a specific price, called *strike price*
- Strike price is $I > 0$
- Question: When is it optimal to stop holding (or *exercise*) the option?
  $\rightarrow$ This is called an **optimal stopping** problem
- Time at which it is optimal to exercise is a random function of time:
  - A stopping time τ is a random variable $\tau : \Omega \rightarrow \mathbb{R}_+$ that is adapted to the relevant filtration
- Problem: $\max_{\tau} \mathbb{E}[e^{-r\tau}(X_\tau - I, 0)^+]$ s.t.
  $dX_t = \mu X_t dt + \sigma X_t dB_t,\ X_0 = x_0 > 0$, where τ is a $\mathbb{F}^X-$stopping time

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Controlled Dynamics and Types of Controls

- Fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and B a $\mathbb{F}-$B.M. that is $d-$dimensional
- $X := (X_t)_{t\geq 0}$ in $\mathbb{R}^n$ denote the agent's relevant state variable:

$$dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dB_t \tag{3}$$

  where $\alpha := (\alpha_t)_{t\geq 0}$ is the control process. For each t, $\alpha_t \in A \subseteq \mathbb{R}^m$

- α is typically $\mathbb{F}-$progressively measurable (a bit more than adapted). Typically three types of controls:
  1. Open loop control control: $\mathbb{F} = \mathbb{F}^B$. That is, there exists $a : [0,T] \times C([0,T]) \to A$ s.t. $a_t(\omega) = a(t, (B_s)_{s\in[0,T]})$ (i.e., $a_t(\omega_1) = a_t(\omega_2)$ when $B_s(\omega_1) = B_s(\omega_2)$ for $s \leq t$)
  2. Closed loop or feedback control: $\mathbb{F} = \mathbb{F}^X$ i.e. $a_t(\omega) = a(t, (X_s)_{s\in[0,T]})$
  3. Markov control: $\mathcal{F}_t = \sigma(X_t)$. That is, $a_t = a(X_t)$

Finite Horizon

- For single-agent models, open loop and feedback coincide and are the most general ones. Assume α is of this form
- Let $\mathcal{A}$ denote the set of all controls α s.t. the previous SDE has a unique solution given any x_0
- **Finite Horizon**: The problem is

$$\max_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t)dt + \psi(X_T)\right]$$

  s.t. the SDE (3), and x_0 is given.

- Given a control α and $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, let

$$\mathcal{L}^u F(t, x) = F_t + b(t, x, u)F_x(t, x) + \frac{1}{2}\mathsf{tr}(\sigma^T(t, x, u)F_{xx}(x, t)\sigma(t, x, u)).$$

Dynamic-Programming Principle

Define the **value function**

$$V(t,y) := \sup_{\alpha \in \mathcal{A}(t,y)} \mathbb{E}\left[\int_t^T f(s, X_s^{t,y}, \alpha_s)\, dt + \psi(X_T) \Big| \mathcal{F}_t\right] \tag{4}$$

where $(X_s^{t,y})_{s \geq t}$ solves (3) with initial value $X_t = y$ and $\mathcal{A}(t,y)$ is the set of feasible strategies for the problem with initial value (t,y).

Theorem

Under mild conditions on f and ψ (typically growth conditions– see Pham (2009)) the continuation value $V(t,y)$ satisfies the equation

$$V(t,y) = \sup_{\alpha \in \mathcal{A}(t,y)} \mathbb{E}\left[\int_0^\tau f(s, X_s^{t,y}, \alpha_s)\, ds + V(\tau, X_\tau^{t,y})\right] \tag{5}$$

where $\tau \in [t, T]$ is an $\mathbb{F}^B-$stopping time.

Hamilton-Jacobi-Bellman Equation

- The **Dynamic-Programming Principle** (DPP) states that the previous problem can be solved in a recursive way
  - But we have to solve a very similar problem an infinite number of times!
  - We would like to obtain something more powerful
- The famous **Hamilton-Jacobi-Bellman** (HJB) equation (presented next) is the *infinitesimal* version of the DPP: it describes the local behavior of the value function when $\tau \to t$
- **Important Result**: If the value function $(t, y) \mapsto V(t, y)$ exhibits enough differentiability ($C^{1,2}$), then it satisfies the HJB equation
  - Nice! We can solve the HJB and in order to find the value function
  - But is the solution to the HJB unique? Next theorem

Verification Theorem for Finite Horizon

Theorem

Let $w : [0, T) \times \mathbb{R}^n \to \mathbb{R} \in C^{1,2}([0, T) \times \mathbb{R}^n)$ *satisfying the growth condition: there exists* $C > 0$*, s.t.* $|w(t, x)| \le C(1 + |x|^2)$ *for all* $(t, x) \in [0, T] \times \mathbb{R}^n$*. Suppose that* w *satisfies the HJB equation*

$$\sup_{a \in A}\{\mathcal{L}^u w(t, x) + f(t, x, a)\} = 0, \ (t, x) \in [0, T) \times \mathbb{R}^n, \tag{6}$$

$$w(T, x) = \psi(x), \ x \in \mathbb{R}^n \tag{7}$$

Then, $w \ge V$ *on* $[0, T] \times \mathbb{R}^n$*. Furthermore, if there is a function* $\hat{\alpha}$ *s.t.* $\hat{\alpha}(t, x)$ *achieves that maximum in the HJB, and s.t.*

$$dX_t = b(t, X_t, \hat{\alpha}(t, X_t))\,dt + \sigma(t, \hat{\alpha}(t, X_t))\,dB_t$$

has a unique solution. Then $w = V$ *and* $\hat{\alpha}$ *is an optimal (Markovian) control.*

Sketch of the Proof

- We will show first that

$$w(t, X_t) \geq \mathbb{E}\left[\int_t^T f(s, X_s, \alpha_s)\, ds + \psi(X_T) \,\Big|(t, X_t)\right],$$

  for any feasible strategy $\alpha \in \mathcal{A}$. Taking $\sup_{a \in \mathcal{A}}$ we conclude that, $w(t, x) \geq V(t, x)$ for all (t, x). Hence w is an upper bound to the agent's attainable utility in the optimization problem

- Finally, we will show that under $\hat{\alpha}$ in the theorem, the above inequality is indeed an equality, so $w(t, x)$ is in fact the agent's value function.

- In particular, $w(0, x_0)$ is the agent's time-zero utility.

Sketch of the Proof: Continued

Define the process

$$W_t^\alpha := \int_0^t f(s, X_s, \alpha_s)\, ds + w(t, X_t),\ t \in [0, T]$$

where $(X_s)_{s \geq t}$ is the dynamic generated by strategy $(\alpha_s)_{s \geq t}$. Observe that $W_0^u = w(0, x_0)$ and

$$\begin{aligned} \mathbb{E}[W_T^\alpha] &= \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t)\, dt + w(T, X_T)\right] \\ &= \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t)\, dt + \psi(X_T)\right] \qquad (8) \\ & \qquad (9) \end{aligned}$$

that is, $\mathbb{E}[W_T^\alpha]$ is the agent's utility when she follows strategy $\alpha \in \mathcal{A}$.

Sketch of the Proof: Continued

- Observe that for a feasible α, $W^\alpha := (W_t^\alpha)_{t\geq 0}$ is an Ito process. As a consequence:

$$dW_t^\alpha = \underbrace{[f(t, X_t, \alpha_t) + \mathcal{L}^\alpha w(t, X_t)]\, dt}_{(*)} + w_x(t, X_t)\sigma(t, X_t, \alpha_t) dB_t$$

where (omitting the dependence of σ on (t, x, α))

$$\mathcal{L}^\alpha w(t, x) = w_t(t, x) + w_x(t, x) b(t, x, \alpha) + \frac{1}{2}\mathrm{tr}(\sigma^T D_{xx}^2 w(t, x)\sigma).$$

- But, since w satisfies the HJB equation, we have that $(*) \leq 0$ a.s. for all $\alpha \in \mathcal{A}$. Thus, W^α is a **supermartingale**. In particular,

$$W_t^\alpha \geq \mathbb{E}[W_T^\alpha | \mathcal{F}_t], \text{ for all} t \in [0, T]$$

Sketch of the Proof: Continued

- But, since $W_t^\alpha := \int_0^t f(s, X_s, \alpha_s)\,ds + w(t, X_t),\ t \geq 0$ we have that

$$\begin{aligned} \mathbb{E}[W_T^\alpha | \mathcal{F}_t] &= \int_0^t f(s, X_s, \alpha_s)\,ds \\ &\quad + \mathbb{E}\left[\int_t^T f(s, X_s, \alpha_s)\,ds + \psi(X_T) \,\middle|\, \mathcal{F}_t \right] \end{aligned} \tag{10}$$

- Hence $W_t^\alpha \geq \mathbb{E}[W_T^\alpha | \mathcal{F}_t],$ for all $t \in [0, T]$ is equivalent to

$$w(t, X_t) \geq \mathbb{E}\left[\int_t^T f(s, X_s, \alpha_s)\,ds + \psi(X_T) \,\middle|\, (t, X_t) \right] \tag{11}$$

for all feasible α. Therefore, $w(t, X_t) \geq V(t, X_t)$ (take $\sup\limits_{\alpha \in \mathcal{A}}$ on the right-side of the above equation).

Sketch of the Proof: Continued

- Now, since there exists $\hat{\alpha}$ that attains the maximum in the HJB, the process $W^{\hat{\alpha}} := (W_t^{\hat{\alpha}})_{t\in[0,T]}$ has zero drift, so it is a martingale.
- In this case, all the above inequalities become equalities, in particular

$$w(t, X_t) = \mathbb{E}\left[\int_t^T f(s, \hat{X}_s, \hat{\alpha}_s)ds + \psi(\hat{X}_T)\ \Big|(t, X_t)\right]$$

  for all (t, X_t), where $(\hat{X}_s)_{s\geq t}$ is the dynamic generated by $\hat{\alpha}$. Since the upper bound can be attained, $V(t,x) = w(t,x)$, so $w(\cdot,\cdot)$ is the value function
- Finally, since $\hat{\alpha} = \hat{\alpha}(t, X_t)$, the optimal control is Markov.

Application: Feynman-Kac

- Suppose there is no optimization problem and interest rate is identically equal to zero. Define

$$V(t,x) = \mathbb{E}\left[\int_t^T f(s, X_s)ds + \psi(x_T) \middle| \, X_t = x\right].$$

- As a consequence, the verification theorem states that if there exists a solution w to the PDE (known as *Cauchy problem*)

$$\cancel{\max_{a \in \mathcal{A}}} \mathcal{L}w(t,x) - w(t,x) = 0, \text{ with } w(T,x) = \psi(x) \tag{12}$$

  that is of class C^2 and satisfies a growth condition, then it must coincide with the function V.

- In particular, the solution to the Cauchy problem is unique and given by $V(t,x)$.

Infinite Horizon

- In order to make the problem stationary consider

$$f(t, x, u) = e^{-rt} h(x, u)$$

$|h|$ growing not too fast (so the value function is finite)

- The state variable evolves as in (3):

$$dX_t = b(X_t, \alpha_t)\, dt + \sigma(X_t, \alpha_t)\, dB_t$$

- Define A and $\mathcal{A}$ as before. Define the continuation value function

$$V(y) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^{\infty} e^{-r(s-t)} f(X_s^{t,y}, \alpha_s)\, ds \middle| \mathcal{F}_t \right],\ X_t = y. \qquad (13)$$

DPP: Infinite Horizon

Theorem

Under integrability and growth conditions on f, the value function $V(y)$ satisfies the equation

$$V(y) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^\tau e^{-rs} f(X_s^y, \alpha_s)\, ds + e^{-r\tau} V(X_\tau^y)\right] \tag{14}$$

where $\tau \in [t, T]$ is an $\mathbb{F}^B-$stopping-time and $(X_s^y)_{s \geq 0}$ is the solution to the controlled SDE

$$dX_t = b(X_t, \alpha_t)\, dt + \sigma(X_t, \alpha_t)\, dB_t$$

when $X_0 = y$.

- Define $\mathcal{L}^u w(x) = b(t, x, u) w_x(x) + \frac{1}{2}\mathsf{tr}(\sigma^T(x, u) w_{xx}(x) \sigma(x, u))$ for $w \in C^2(\mathbb{R})$

Verification Thm and HJB for Infinite Horizon

Theorem

Let $w \in C^2(\mathbb{R}^n)$ satisfying a quadratic growth condition be the solution to the HJB equation

$$\begin{aligned} \sup_{a \in A}\{\mathcal{L}^a w(x) + f(x,u)\} &= rw(y),\ x \in \mathbb{R}^n, &(15)\\ \lim_{T\to\infty} e^{-rT}\mathbb{E}[w(X_T^x)] &= 0,\ \forall x \in \mathbb{R}^n,\ \forall \alpha \in \mathcal{A}(x). &(16) \end{aligned}$$

Then, $w \geq v$ on $\mathbb{R}^n$. Moreover, if there exists $\hat{\alpha} : \mathbb{R} \to A$ such that (i) for all $x \in \mathbb{R}$, $\hat{\alpha}(x)$ attains the maximum in the HJB and (ii) the SDE

$$dX_t = b(X_t, \hat{\alpha}(X_t))\,dt + \sigma(X_t, \hat{\alpha}(X_t))\,dB_t,\ X_0 = x,$$

admits a unique solution $(\hat{X}_t^x)_{t\geq 0}$, and $\lim_{T\to\infty} e^{-rT}\mathbb{E}[w(\hat{X}_T^x)] = 0$, for all $x \in \mathbb{R}$, then $w = v$ and $\hat{\alpha}(\cdot)$ is an optimal Markovian control.

Example 1: Merton's Portfolio Problem – Finite Horizon

Optimal Portfolio and consumption:

$$\max_{(\alpha,C)} \quad \mathbb{E}\left[\int_0^T u(t,c_t)\,dt + \psi(W_T)\right] \tag{17}$$

$$s.t. \quad \frac{dW_t}{W_t} = \left(\alpha\cdot(\mu - r\vec{e}) + r - \frac{c_t}{W_t}\right)dt + \alpha\cdot\sigma\,dZ_t \tag{18}$$

$$W_0 > 0 \tag{19}$$

HJB:

$$\sup_{c,\alpha}\{\,V_t(W,t) + V_w(W,t)(W\alpha^T(\mu - r\vec{e}) + Wr - c) \tag{20}$$

$$+\frac{1}{2}W^2\alpha^T\sigma\sigma^T\alpha V_{ww}(W,t) + u(c,t)\} = 0 \tag{21}$$

Merton's Problem– Finite Horizon: Continued

- FOC:

$$\alpha = -\frac{V_w(w,t)}{wV_{ww}(w,t)}(\sigma\sigma^T)^{-1}(\mu - r\vec{e}) \qquad (22)$$

$$u_c(c,t) = V_w(w,t) \qquad (23)$$

- Special case: $u(c,t) = 0$, $c \geq 0$ and $\psi(W) = W^\gamma/\gamma$, $\gamma < 1$
- It is easy to see that if a level of final wealth Z_T is optimal when wealth starts from $W_0 = 1$, then wZ_T is optimal if we start from $W_0 = w$. **Question:** Can you show that?
- Hence, if Z denotes the optimal (unknown) final wealth if $w = 1$:

$$V(0,w) = \mathbb{E}[(wZ)^\gamma/\gamma] = \frac{w^\gamma}{\gamma}\mathbb{E}[Z^\gamma]$$

- Conjecture: $V(t,w) = \frac{w^\gamma}{\gamma}k(t)$. Need to find f.

Merton's Problem– Finite Horizon: Continued

- Plug the conjecture into the FOC: $\frac{V_w}{wV_{ww}} = -\frac{1}{1-\gamma}$, so

$$\alpha = \frac{(\sigma\sigma^T)^{-1}(\mu - r\vec{e})}{1-\gamma}$$

- Plug this expression and the conjecture $V(t,w) = w^\gamma/\gamma k(t)$ in the HJB. It reduces to

$$k'(t) + bk(t) = 0$$

where $b = \frac{\gamma(\mu - r\vec{e})^T(\sigma\sigma^T)^{-1}(\mu - r\vec{e})}{2(1-\gamma)} + r\gamma$, and boundary condition $f(T) = 1$ (why?)

- Solving the previous ODE, $k(t) = e^{b(T-t)}$, so $V(t,w) = \frac{w^\gamma e^{b(T-t)}}{\gamma}$

Merton's Portfolio Problem – Infinite Horizon

- $u(t, c) = e^{-\rho t}\frac{c^\gamma}{\gamma}$ and $\psi \equiv 0$
- Conjecture: $V(w) = K\frac{w^\gamma}{\gamma}$;
- FOC:

$$c_t = \theta W_t,\ \alpha_t = \frac{(\sigma\sigma^T)^{-1}(\mu - r\vec{e})}{1-\gamma} \tag{24}$$

where $\theta = K^{\frac{1}{\gamma-1}}$. Plug these expressions the conjectured value function into the HJB jointly. We obtain:

$$\theta = \frac{\rho - r\gamma}{1-\gamma} - \frac{\alpha(\mu - r\vec{e})^T(\sigma\sigma^T)^{-1}(\mu - r\vec{e})}{2(1-\alpha)^2} \tag{25}$$

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Stopping Time Problems

- Sometimes we want to treat T as a control variable
  - Since the decision to stop depends on a state variable that is random $\Rightarrow$ likely that the time to stop is random (i.e., a stopping time)
  - Exercise the call option if and only if the price of the stock is above $45
- Classic examples:
  - (i) Optimal exercise of an American option
  - (ii) Optimal investment timing when investing is an irreversible decision (real option)
- **Solution**: involve both a continuation region and a stopping region depending on the values of the state variable
  - In the continuation region we have a standard HJB equation
  - In the stopping region the process is immediately stopped $\Rightarrow$ value function takes the value of the "outside option"

Example: Optimal Exercise of American Call Option

- Stock price evolves as

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \; X_0 = x_0 > 0; \tag{26}$$

- Infinitely-lived American call: right (but not the obligation) to buy at a specific price. Denote the **strike** price by $I \geq 0$
- Problem: $\max_{\tau} \mathbb{E}[e^{-r\tau}(X_\tau - I)^+]$ s.t. (26) where τ is a $\mathbb{F}^X-$stopping time.
- By stationarity, the **value of the option** depends only on x and it is given by

$$V(x) := \max_{\tau} \mathbb{E}[e^{-r\tau}(X_\tau - I)^+ | X_0 = x]$$

- Observe that $V(x) \geq (x - I)^+$ for all $x \geq 0$ since it is always feasible to exercise immediately.

Example: Continuation and Stopping Regions

- Stationary problem $\Rightarrow$ natural conjecture is that it is optimal to exercise immediately as the stock price reaches a specific threshold
  - Equivalently, since the stock price evolves stochastically, for low values of the stock it is optimal to wait for it to go up
- Define a stopping region as $\mathcal{S} = \{x \in \mathbb{R} | \ x \geq \overline{x}\}$ and a continuation region as $\mathcal{C} = \{x \in \mathbb{R} | \ x < \overline{x}\}$
- Two questions: 1) Is this proposed method (two regions of this form) the solution to the stopping problem? 2) If this is correct, what is the optimal threshold $\overline{x}$?
- Assume the solution involves regions of this form and then verify

Example: Arbitrage Equation

- Clearly, $\overline{x} \geq I$ (one has always the choice to give the option away, which yields a payoff of zero)
- Thus, in the **stopping region** (i.e., $x \geq \overline{x}$) $V(x) = x - I \geq 0$
- In the **continuation region** the option is not exercised and hence no cash flows are received. The value of the option $V(X_t)$ evolves freely as the price of the underlying asset X_t changes over time. Thus, the following equation must hold:

$$rV(x) = V_x(x)\mu x + \frac{1}{2}V_{xx}\sigma^2 x^2 \tag{27}$$

- Do you recall this arbitrage equation?

Example: Arbitrage Equation

- Mathematically, observe that by the DPP and dt small:

$$\begin{aligned} V(X_t) &= 0dt + e^{-rdt}\mathbb{E}[V(X_{t+dt})|\,X_t] \\ &= (1-rdt)\mathbb{E}[V(X_t) + V_x(\mu X_t dt + \sigma dB_t) \\ &\qquad\qquad + \frac{1}{2}V_{xx}(X_t)\sigma^2(X_t^2)dt|\,X_t] \\ \Rightarrow \cancel{V(X_t)} &= \cancel{V(X_t)} + [V_x\mu X_t + \frac{1}{2}V_{xx}(X_t)\sigma^2(X_t^2) - rV(X_t)]dt \end{aligned}$$

  where in the last equality we ignore all the $o(dt^2)$ terms.

- The general solution of (27) is given by $V(x) = Ax^\alpha + Bx^\beta$ where $\alpha > 1$ and $\beta < 0$ solve

$$\frac{1}{2}\sigma^2\alpha(\alpha-1) + \alpha\mu - r = 0$$

.

Example: Value-Matching Condition

- Observe that if $X = 0$ then $X_t \equiv 0$ a.s. for all t (geometric B.M.!) $\Rightarrow$ the option is worthless. In that case $V(0) \equiv 0$ which implies that $B = 0$ (otherwise V would explode)
- Hence, in the continuation region $V(x) = Ax^\alpha$, $\alpha > 0$. Find A?
- But at the threshold $x = \overline{x}$ the agent must be indifferent between exercising the option or holding it. Hence, we impose

$$A\overline{x}^\alpha = \overline{x} - I \Rightarrow A(\overline{x}) = \frac{\overline{x} - I}{\overline{x}^\alpha} > 0$$

which is called the value-matching condition

- Still need to find $\bar{x}$! Need an additional equation...

Example: Smooth-Pasting Condition

- Smooth pasting condition: The derivatives of V and $(x - I)$ should coincide at $\overline{x}$:

$$V'(\overline{x}) = 1 \Leftrightarrow A\alpha\overline{x}^{\alpha-1} = 1$$

- Solving the system yields:

$$\begin{aligned} V(x) &= \frac{I}{\alpha-1}\left(\frac{\alpha I}{\alpha-1}\right)^{-\alpha} x^{\alpha} \text{ if } 0 < x < \frac{\alpha I}{\alpha-1} \\ &= x - I, \text{ if } x \geq \frac{\alpha I}{\alpha-1} \end{aligned} \tag{28}$$

- Intuition for the smooth pasting condition?
  - $V'(\overline{x}) > 1$ implies $x - I > V(x)$ for x slightly to the left of $\overline{x}$, violating the condition that $V(x) \geq (x - I)^{+}$ (i.e. it would be optimal to exercise earlier)
  - If instead $V'(\overline{x}) < 1$ then the agent would be better off by waiting a bit beyond $\overline{x}$ to the right than stopping at $\overline{x}$ – next.

$V'(\overline{x}) < 1 \Rightarrow$ Waiting beyond $\overline{x}$ is Optimal

- Payoff of not exercising at $\overline{x}$? Consider the binomial approximation of the geometric B.M.: Δ small unit of time and

$$X_{t+\Delta} = X_t + \underbrace{\sigma\sqrt{\Delta}}_{h:=}, \text{ with prob } \; p = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right]$$

and $X_{t+\Delta} = X_t - \sigma\sqrt{\Delta}$ with prob. $1-p$

- Consider alternative strategy of exercising at $\overline{x} + h$. Then:

$$\begin{aligned}
\tilde{V} &= 0 + (1 - r\Delta)[p\underbrace{(\overline{x} + h - I)}_{\text{exercise}} + (1-p)\underbrace{V(\overline{x} - h)}_{\text{hold the option}}] \\
&= (1 - r\Delta)[p(\overline{x} + h - I) + (1-p)\{V(\overline{x}) - V'(\overline{x} - h)h + ...\}] \\
&= (1 - r\Delta)\left[V(\overline{x}) + p\{(\overline{x} - I) - V(\overline{x})\} + \sigma\sqrt{\Delta}\{p - (1-p)V'(\overline{x})\}\right] \\
&\quad + ... \qquad (29)
\end{aligned}$$

$V'(\overline{x}) < 1 \Rightarrow$ Waiting beyond $\overline{x}$ is Optimal

$$\begin{aligned}
\tilde{V} &= (1 - r\Delta)\,V(\overline{x}) + (1 - r\Delta)p\underbrace{\{(\overline{x} - I) - V(\overline{x})\}}_{=0,\text{ value matching}} \\
&\quad + (1 - r\Delta)\sigma\sqrt{\Delta}\{p + (1-p)\,V'(\overline{x})\} + (1 - r\Delta)(1-p)\underbrace{[O(\Delta)]}_{small}\ldots \\
&= \underbrace{V(\overline{x})}_{=\overline{x}-I} + \sigma\sqrt{\Delta}\{p + (1-p)\,V'(\overline{x})\} \qquad (30)
\end{aligned}$$

where in the last equality we have neglected all the terms of order Δ or higher. But since $V'(\overline{x}) < 1 \Rightarrow$, $p + (1-p)\,V'(\overline{x}) \approx 1/2[1 - V'(\overline{x})] > 0$ for Δ small. Hence,

$$\tilde{V} > \overline{x} - I \qquad (31)$$

So, waiting further above $\overline{x}$ is a better strategy. Hence, if $\bar{x}$ is optimal, smooth pasting must hold.

Finally the Theorem

- Consider the problem $\sup_\tau \mathbb{E}[e^{-\tau}\psi(X_\tau)]$, where $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$ and the associated **value function** $V(x) := \sup_\tau \mathbb{E}[e^{-\tau}\psi(X_\tau)|X_0 = x]$.

Theorem

Suppose the w is a function of class C^2 with the following properties:

- $w(x) \geq \psi(x)$ *for all* $x \in \mathbb{R}$*,*
- $0 \geq -rw(x) + w'(x)\mu(x) + \frac{1}{2}\sigma^2(x)w''(x)$ *with equality over the set* $C := \{x \mid w(x) > \psi(x)\}$ *(cont. region).*

Then $w(x) \geq V(x)$ *for all* $x \in \mathbb{R}$*. Let* $S := \{x \mid w(x) = \psi(x)\}$ *(stopping region) and suppose that* $\tau^* := \inf\{t : \ w(x) = \psi(x)\} < \infty$ *a.s. Then* τ^* *is an optimal stopping time.*

Obs: The result can be generalized to problems that allow for both flow payoffs and a diffusion X that is controlled by the agent.

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