Two-Sided Learning and the Ratchet Principle

Gonzalo Cisternas
MIT Sloan

This Paper

- Class of games of learning and imperfect monitoring. Key elements:
  - **Learning** about a hidden state
  - **Ex ante symmetric uncertainty**
  - **Imperfect monitoring** $\rightarrow$ strategically affecting beliefs of others
- Challenge: **off-path private beliefs**. Existing literature:
  - Reputation: linear payoffs (Holmström, 1999; Board and Meyer-ter-Vehn 2013 and 2014)
  - Experimentation: coarse information structures (Bergemann and Hege, 2005; Bonatti and Horner 2010 and 2017)
- This paper: framework that accommodates (i) non-linear payoffs and (ii) frequent arrival of information; no experimentation effects

Model

- Long-run player (LRp) and continuum of small players (*market*)
- $t \in [0, \infty)$; partially observed process

$$\begin{aligned} d\theta_t &= -\kappa(\theta_t - \eta)dt + \sigma_\theta dZ_t^\theta \qquad \textit{fundamentals - hidden} \\ d\xi_t &= (a_t + \theta_t)dt + \sigma_\xi dZ_t^\xi \qquad \textbf{public signal} \end{aligned}$$

  - Common prior: $\theta_0 \sim \mathcal{N}(p, \gamma^*)$; $a_t \in A \subset \mathbb{R}$ **LRp's hidden action** $t \geq 0$

- Market only observes $(\xi_t)_{t\geq 0}$ and takes action $\chi(a_t^*, p_t^*)$
  - $(a_t^*)_{t\geq 0}$: mkt's conjecture of behavior; $p_t^* := \mathbb{E}^{a^*}[\theta_t | (\xi_s)_{s\leq t}]$

- Given a^*, LRp chooses $(a_t)_{t\geq 0}$ to maximize

$$\mathbb{E}^{a}\left[\int_0^\infty e^{-rt}\left(u(\chi(a_t^*, p_t^*)) - g(a_t)\right)dt\right]$$

Today: $g(a) = a^2/2$ - can be more general ▸ Diff. & growth conditions

Reputation in Labor Markets: Career Concerns (Holmström 1999)

- Worker and a pool of employers; a_t: worker's **hidden effort**

$$
\begin{aligned}
d\theta_t &= -\kappa(\theta_t - \eta)dt + \sigma_\theta dZ_t^\theta && \textbf{skills - hidden} \\
d\xi_t &= (a_t + \theta_t)dt + \sigma_\xi dZ_t^\xi && \textbf{output - public}
\end{aligned}
$$

- Labor market is *spot* (competition & no explicit contracts):

$$
\textbf{wage at t} := \lim_{h \to 0} \frac{\mathbb{E}^{a^*}[\xi_{t+h} - \xi_t]}{h} = p_t^* + a_t^* =: \chi(a_t^*, p_t^*)
$$

- Risk neutral worker ($u(\chi) = \chi$) chooses $(a_t)_{t \geq 0}$ to maximize

$$
\mathbb{E}^a \left[\int_0^\infty e^{-rt} \left(p_t^* + a_t^* - \frac{a_t^2}{2} \right) dt \right]
$$

Equilibrium effort? Key: flow payoff is linear in p_t^*

Macroeconomics: Monetary Policy

- Central bank and an economy; a_t: **money growth**

$$\begin{aligned} d\theta_t &= -\kappa\theta_t dt + \sigma_\theta dZ_t^\theta && \textbf{inflation trend - hidden} \\ d\xi_t &= (a_t + \theta_t)dt + \sigma_\xi dZ_t^\xi && \textbf{log price index - public} \end{aligned}$$

- *Phillips curve*: log employment (n) evolves according to

$$dn_t = -\kappa_n n_t dt + \nu \underbrace{[d\xi_t - (a_t^* + p_t^*)dt]}_{\text{unanticipated inflation; } dS_t}$$

- Changes in p_t^* also driven by $dS_t \Rightarrow n_t = p_t^*$ some parameters

- Central bank chooses $(a_t)_{t\geq 0}$ to maximize

$$\mathbb{E}^a \left[\int_0^\infty e^{-rt} \left(-\frac{n_t^2}{2} - \frac{a_t^2}{2} \right) dt \right]$$

Equilibrium inflation? Flow payoff **nonlinear** in $n_t = p_t^*$

Financial Markets: Earnings Management

- Manager and a financial market; a_t: **earnings manipulation**

$$\begin{aligned} d\theta_t &= \sigma_\theta dZ_t^\theta && \textbf{firm's fundamentals - hidden} \\ d\xi_t &= (a_t + \theta_t)dt + \sigma_\xi dZ_t^\xi && \textbf{earnings report - public} \end{aligned}$$

- Market expects true earnings $\mathbb{E}^{a^*}[d\xi_t - a_t dt] = p_t^* dt$ over $[t, t+dt)$
- Manager's incentives are more acute at $p^* = 0$ (*zero earnings threshold*): $\chi' > 0$ and maximized at zero
- Manager chooses $(a_t)_{t\geq 0}$ to maximize

$$\mathbb{E}^a\left[\int_0^\infty e^{-rt}\left(\chi(p_t^*) - \frac{a_t^2}{2}\right)dt\right]$$

Equilibrium policy $a^*(p^*)$? Flow payoff is **fully nonlinear**

Overview of Results

1. Main finding: learning-driven "ratchet principle"
2. Technical contribution: "first-order approach" (FOA) to perform equilibrium analysis with off path hidden actions + hidden info

Specifically:

- ODE as a necessary condition for Markov equilibrium $a_t^* = a^*(p_t^*)$
  - *ratcheting equation*: "Euler equation + ratchet-like forces" $\rightarrow$ novel
- Sufficiency: **verification theorem**
  - bypasses belief divergence/private beliefs challenge
- Existence of pure-strategy equilibria in environments with **nonlinear** flow payoffs
- Applications: ratchet effects & nonlinearities

Roadmap

1. Strategies and Equilibrium Concept
2. Laws of Motion of Beliefs
3. Necessary Condition: Ratcheting Equation
4. Applications
5. Sufficiency: Verification Theorem
6. Existence of Markov Equilibria

Public Strategies and Nash Equilibrium

- $(\xi_t)_{t\geq 0}$ satisfies the full-support assumption
  - "Nash eq. is outcome-equivalent to sequential eq." $\Rightarrow$ focus on **Nash**

- A *pure strategy* $(a_t)_{t\geq 0}$ is **feasible** if it is ξ-prog. measurable, $\mathbb{E}[\int_0^t (a_s^2)ds] < \infty$, and $(\xi_t)_{t\geq 0}$ has a solution

- **Nash eq.**: $(a_t^*)_{t\geq 0}$ is optimal for the LRp when $p_t^* = p_t^*[\xi, a^*]$, $t \geq 0$
  - Belief coincide on the equilibrium path

- A N.E. is **Markov** if $a_t^* = a^*(p_t^*)$, $a^* \in C^2(\mathbb{R}; A)$, Lipschitz

In what follows, mkt's conjecture $(a_t^*)_{t\geq 0}$ is fixed and study deviations

Law of Motion of Beliefs

- $d\xi_t - a_t dt = \underbrace{\theta_t dt + \sigma_\xi dZ_t^\xi}_{dY_t :=}$; $p_t := \mathbb{E}[\theta_t | (Y_s)_{0\leq s\leq t}]$: **LRp's belief**
- Prior variance $\gamma^* := \sigma_\xi^2[(\kappa^2 + \sigma_\theta^2/\sigma_\xi^2)^{1/2} - \kappa] \Rightarrow$ posteriors $\mathcal{N}(\cdot, \gamma^*)$

Lemma

$$dp_t^* = -\kappa(p_t^* - \eta)dt + \underbrace{\frac{\gamma^*}{\sigma_\xi^2}}_{\color{blue}\beta :=}[d\xi_t - (a_t^* + p_t^*)dt] \quad \textit{and} \tag{1}$$

$$dp_t = -\kappa(p_t - \eta)dt + \underbrace{\frac{\gamma^*}{\sigma_\xi}}_{\color{blue}\sigma :=} \underbrace{\frac{[d\xi_t - (a_t + p_t)dt]}{\sigma_\xi}}_{= dY_t - p_t := {\color{blue}dZ_t}}, \; t \geq 0, \tag{2}$$

where $(Z_t)_{t\geq 0}$ *is a BM from the LRp's perspective and* $d\xi_t = (a_t + p_t)dt + \sigma_\xi dZ_t$ *from his standpoint.*

Obs: $a_t > a_t^* \Rightarrow p_t^* > p_t$, i.e., **off-path belief asymmetry**

HJB Approach

Theorem

Suppose that V satisfies the equation

$$rV(p,p^*) = \sup_{a\in A}\Bigg\{ \underbrace{u(\chi(p^*, a^*(p^*))) - a^2/2}_{\textit{flow payoff}} \underbrace{-\kappa(p-\eta)}_{\textit{drift of } p} V_p(p,p^*)$$

$$+ \underbrace{[-(\beta+\kappa)(p^*-p) + \beta[a - a^*(p^*)]]}_{\textit{drift of } p^*} V_{p^*}(p,p^*)$$

$$+ \underbrace{0.5\sigma^2[V_{pp}(p,p^*) + 2V_{p,p^*}(p,p^*) + V_{p^*p^*}(p,p^*)]}_{\textit{second-order terms}}\Bigg\}$$

$$s.t. \qquad \arg\max_{a\in A}\{\beta V_{p^*}(p^*,p^*)a - a^2/2\} = a^*(p^*)$$

Then $a^*(p^*) = \beta V_{p^*}(p^*,p^*)$ *is a* ***Markov eq.*** *and* $V(p,p)$ *its payoff.*

- Non-local PDE: $V_{p^*}(p^*,p^*)$ term

Necessary Conditions and Ratchet Principle

Ratchet Principle

- Suppose $\chi(p^*, a^*) = p^*$; $(p_t^*)_{t \geq 0}$ is effectively an *incentive scheme*

$$\underbrace{dp_t^*}_{\text{change in payments}} = \underbrace{-\kappa(p_t^* - \eta)dt}_{\text{exogenous trend}} + \underbrace{\beta}_{\text{sensitivity}} \times [\underbrace{d\xi_t}_{\text{performance}} - \underbrace{(p_t^* + a^*(p_t^*))dt}_{\text{target performance}}].$$

- Ratchet:= sensitivity of target to contemporaneous performance

$$:= \frac{d(p_t^* + a^*(p_t^*))}{d\xi_t} = \left[1 + \frac{da^*(p^*)}{dp^*}\right]\Bigg|_{p^*=p_t^*} \times \underbrace{\frac{dp_t^*}{d\xi_t}}_{=\beta} = \beta + \beta\frac{da^*(p_t^*)}{dp^*}$$

- Holds for general χ via Ito's rule

Ratcheting Cost - Heuristic

- Deviation: LRp chooses $a_t = a(p_t^*) + 1$ and matches target thereafter

$$\underbrace{dp_t^*}_{\text{change in payments}} = \underbrace{-\kappa(p_t^* - \eta)dt}_{\text{exogenous trend}} + \underbrace{\beta}_{\text{sensitivity}} \times [\underbrace{d\xi_t}_{\text{performance}} - \underbrace{(p_t^* + a^*(p_t^*))dt}_{\text{target performance}}].$$

- 1) $\Rightarrow p_{t+dt}^* - p_{t+dt} > 0$
- 2) $\Rightarrow a_t = a^*(p_t^*) + p_t^* - p_t \Rightarrow d\xi_s - (a^*(p_s^*) + p_s^*)ds =$ martingale
- 1)+2) $\Rightarrow$ payments increase by $p_s^* - p_s = e^{-\kappa(s-t)} > 0$

- Matching market's expectation of performance is costly:

$$\underbrace{g(a_t) - g(a^*(p_s))}_{\text{extra cost}} = g'(a^*(p_s)) \times \underbrace{\left[1 + \frac{da^*(p_s)}{dp^*}\right]}_{\text{ratchet}} \beta\, e^{-\kappa(s-t)}$$

Necessary Condition

Proposition (**Necessary Conditions**)

In a Markov equilibrium, $a^*(p) = \beta q(p)$*, where*

$$q(p) := \mathbb{E}\left[\int_0^\infty e^{-(r+\kappa)t}\left[(u\circ\chi)'(p_t) - g'(a^*(p_t))\left(1 + \frac{da^*(p_t)}{dp^*}\right)\right]dt \middle| p_0 = p\right]$$

and $dp_t = -\kappa(p_t - \eta)dt + \sigma dZ_t$*,* $p_0 = p$*. The corresponding equilibrium payoff is given by*

$$U(p) := \mathbb{E}\left[\int_0^\infty e^{-rt}[u(\chi(p_t)) - g(\beta q(p_t))]dt \middle| p_0 = p\right].$$

$\rightarrow q(\cdot)$ is a measure of marginal utility in which future beliefs differ

ODEs

Since $g(a) = a^2/2$ and $g'(a^*(p)) = \beta q(p) \Rightarrow da^*(p)/dp = \beta q'(p)$

Proposition (**ODE Characterization**)

In a Markov equilibrium, $a^*(\cdot) = \beta q(\cdot)$, *where* $q(\cdot)$ *satisfies*

$$\Big[r + \kappa + \underbrace{\beta + \beta^2 q'(p)}_{\text{ratchet}}\Big] q(p) = (u \circ \psi)'(p) - \kappa(p-\eta) q'(p) + \frac{1}{2}\sigma^2 q''(p)$$

$U(\cdot)$ *in turn satisfies the linear ODE*

$$rU(p) \quad = \quad u(\chi(p)) - g(\beta q(p)) - \kappa(p-\eta)U'(p) + \frac{1}{2}\sigma^2 U''(p)$$

Ratcheting ODE

$$\Big[\underbrace{r+\kappa}_{\text{discounting}} + \underbrace{\beta+\beta^2 q'(p)}_{\text{ratchet}}\Big] q(p) = \underbrace{(u\circ\psi)'(p)}_{\text{myopic}} \underbrace{-\kappa(p-\eta)q'(p)+\frac{1}{2}\sigma^2 q''(p)}_{\text{cost smoothing}}$$

- Ratchet puts **downward** pressure on incentives. Ratcheting costs
  - $\beta q(\cdot)$ results from off-path belief asymmetry (which matters for on-path incentives!) $\rightarrow$ not borne along the path of play
  - $\beta \frac{da^*(\cdot)}{dp^*} q(\cdot)$ results from changes in a^* $\rightarrow$ borne along the path of play

- Interaction term $q'(\cdot)q(\cdot)$ has opposite effect in eqns. for mg. utility in **decision problems**; but this is a **game**

Applications:

Impact of β and $\beta \frac{da^*}{dp^*}$ on incentives

Career Concerns

$$\Big[r+\kappa+\beta+\beta^2 q'(p)\Big]q(p) = [u(\psi(p,\beta q(p)))]' - \kappa(p-\eta)q'(p) + \frac{1}{2}\sigma^2 q''(p)$$

- $u(\chi)=\chi$, and wage $=\chi(p^*,a^*) = p^* + a^* = p^* + \beta q(p^*)$

$$\Rightarrow [u(\psi(p,\beta q(p^*)))]' = 1 + \beta q'(p)$$

- Look for constant q:

$$q = \frac{1}{r+\kappa+\beta} \Rightarrow g'(a^*(p)) = \frac{\beta}{r+\kappa+\beta}$$

as ratchet $=\beta + \beta\frac{da^*}{dp} = \beta$ in a deterministic equilibrium

- Intuition: recall $dp_t^* = \kappa(p_t^*-\eta)dt + \beta[d\xi_t - (a^*(p_t^*) - p_t^*)dt]$; consider one-time mg. surprise along the path of play vs. deviation

Monetary Policy

- Central bank chooses $(a_t)_{t\geq 0}$ to maximize

$$\mathbb{E}^a\left[\int_0^\infty e^{-rt}\left(-\frac{n_t^2}{2}-\frac{a_t^2}{2}\right)dt\right]$$

where $dn_t = -\kappa n_t dt + \beta[d\xi_t - (a_t^* + p_t^*)dt]$, $n_0 = p_0 \Rightarrow n_t = p_t^*$

- Suppose $(\theta_t)_{t\geq 0}$ is observable or absent. Then

$$dn_t = [-\kappa n_t + \beta(a_t - a_t^*)]dt + \sigma dZ_t^\xi$$

$\Rightarrow$ environment becomes one of imperfect monitoring only

Proposition (Observable case)

In any linear equilibrium, $a^{*,o}(n) = \beta\alpha^o n$*, where* $\alpha^o < 0$.

Monetary Policy

- Hidden case: Phillips curve now becomes

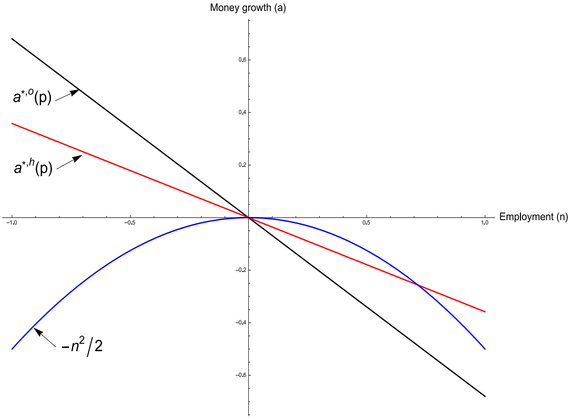
$$dn_t = [-\kappa n_t + \beta(a_t - a_t^*) - \underbrace{\beta(p_t^* - p_t)}_{\text{ratcheting}}]dt + \sigma dZ_t$$

  But $a_t > a_t^* \Rightarrow p_t^* > p_t$, i.e. ratcheting puts **extra downward** pressure on employment

Proposition (Hidden case)

There exists a linear eqbm. $a^{*,h}(n) = \beta\alpha^h n$, $\alpha^h < 0$, *s.t.* $|\alpha^h| < |\alpha^o|$.

Monetary Policy



- Ratcheting generates *commitment* & lower inflationary bias for $n < 0$

$$\underbrace{\left[r + 2\kappa + \beta^2\alpha^o\right]\alpha^o = -1}_{\text{obs. case}} \quad \text{and} \quad \underbrace{\left[r + 2\kappa + \beta + \beta^2\alpha^h\right]\alpha^h = -1}_{\text{hidden case}},$$

Earnings Manipulation

- Recall

$$\begin{aligned} d\theta_t &= \sigma_\theta dZ_t^\theta && \textbf{firm's fundamentals} \\ d\xi_t &= (a_t + \theta_t)dt + \sigma_\xi Z_t^\xi && \textbf{earnings report} \end{aligned}$$

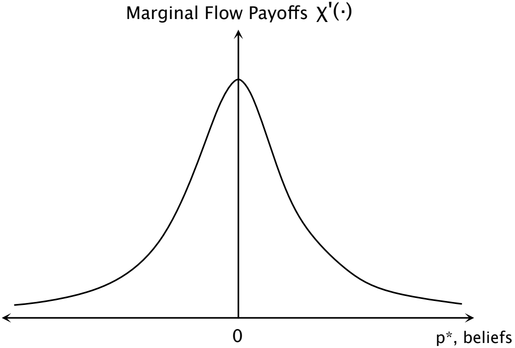
- Market expects true earnings $\mathbb{E}^{a^*}[d\xi_t - a_t^* dt | \mathcal{F}_t] = p_t^* dt$
- Manager's flow payoff: $u(\chi(p^*)) - g(a) = \chi(p^*) - a^2/2$
- Evidence: manipulation strong at key **thresholds** or **benchmarks**
  - zero earnings, zero earnings growth, and/or analysts' forecasts
  - Burgstahler and Dichev (1997); Degeorge at al. (1999); Burgstahler and Chuck (2012); Dichev et al. (2013)

Thresholds and Nonlinear Incentives

- Recall $\mathbb{E}_t^{a^*}[d\xi_t - a_t^* dt] = p_t^* dt$; $p^* = 0$: *zero-earnings threshold*

Assumption

$\chi'(\cdot) > 0$ *is single-peaked and symmetric around zero, with* $\chi'''(0) < 0$

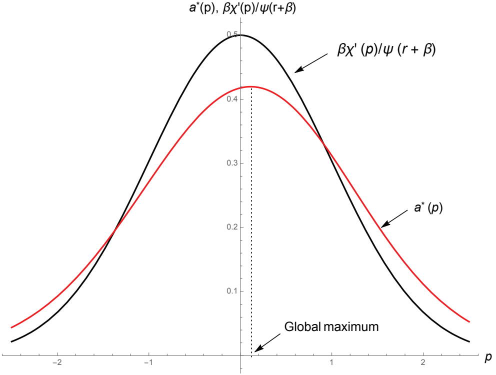


- $p \mapsto \beta\chi'(p)/(r+\beta)$ **myopic benchmark**

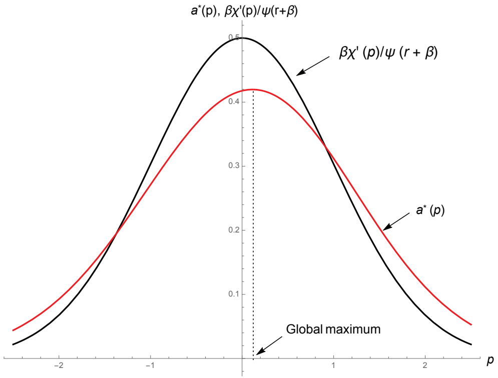
Equilibrium: Policy Skewed to the Right

Proposition

In equilibrium: $a^*(0) < \frac{\chi'(0)}{r+\beta}$, $\frac{da^*}{dp^*}(0) > 0$ *and* $a^*(p) > a^*(-p),\ \forall p > 0$



Endogenous Ratcheting



- Ratcheting $\beta da^*/dp^*$ breaks the symmetry:

$$q(p) = \frac{\chi'(p) + \frac{1}{2}\sigma^2 q''(p)}{r + \beta + \beta^2 q'(p)} \quad = \quad \frac{\chi'(p) + \frac{1}{2}\sigma^2 q''(p)}{r + \beta + \beta \frac{da^*}{dp}(p)}$$

Sufficiency

$$
\begin{aligned}
\left[r + \beta + \kappa + \beta^2 q'(p)/\psi\right] q(p) &= (u \circ \chi)'(p) - \kappa(p - \eta)q'(p) + \frac{1}{2}\sigma^2 q''(p) \\
rU(p) &= (u \circ \chi)(p) - g(\beta q(p)/\psi) - \kappa(p - \eta)q'(p) \\
&\quad + \frac{1}{2}\sigma^2 U''(p)
\end{aligned}
$$

Verification Theorem

Theorem (General $g(\cdot)$ with $g'' > \psi > 0$)

(U, q) of class C^2 solves the previous system + TVC and satisfies

$$|U''(\cdot) - q'(\cdot)| \leq \frac{\psi(r + 4\beta + 2\kappa)}{4\beta^2}. \quad (*)$$

If $a_t^* = (g')^{-1}(\beta q(p_t^*[\xi]))$ is feasible, $a^*(\cdot)$ is a Markov equilibrium and $U(\cdot)$ its corresponding payoff.

- (*) is a bound on the rate of change of **information rent**, $q - U'$
- Idea: construct upper bound to $V(p, p^*)$ that coincides with U on the equilibrium path
  - Use info rent $U' - q$ to account for the value of having pvte information about ξ ▸ Bound

Existence of Markov Equilibria

$$\chi = \chi(p^*) \;\; \& \;\; g'' > \psi$$

Linear-Quadratic Games

Definition (LQ Environment)

$A = \mathbb{R}$; $g(a) = \frac{\psi}{2}a^2$, $\psi > 0$; and $u(\chi(p^*, a^*)) = u_0 + u_1 p^* - u_2 p_t^{*2}$, where $u_0, u_1 \in \mathbb{R}$ and $u_2 \geq 0$.

Theorem

A linear $q(\cdot)$ and a quadratic $U(\cdot)$ solving (q, U) system exist iff

$$u_2 \leq \frac{\psi(r + \beta + 2\kappa)^2}{8\beta^2}.$$

In this case, $a^*(p) = \beta[q_1 + q_2 p]/\psi$, with $q_1 = \frac{\eta\kappa q_2 + u_1}{r + \beta + \kappa + \frac{\beta^2}{\psi} q_2}$ and

$$q_2 = \frac{\psi}{2\beta^2}\left[-(r + \beta + 2\kappa) + \sqrt{(r + \beta + 2\kappa)^2 - \frac{8u_2\beta^2}{\psi}}\right] < 0,$$

is a linear Markov equilibrium.

Bounded Marginal Flow Payoffs

(i) $h(p) := u(\chi(p))$ is differentiable and $m := \inf_{p \in \mathbb{R}} h'(p) > -\infty$ and $M := \sup_{p \in \mathbb{R}} h'(p) < \infty$

(ii) $g : A \to \mathbb{R}$ is twice differentiable and strongly convex, and $g^{-1}(J) \subset A$, where $J := \left[\frac{m}{r+\beta(\kappa)+\kappa}, \frac{M}{r+\beta(\kappa)+\kappa}\right]$

A solution to the ratcheting equation is *bounded* if q and q' are bounded

Existence of Markov Equilibria

Theorem

- *There exists a bounded solution* $q(p)$ *to the ratcheting eqn. taking values in* J
- *Given* $q(p)$*, there exists a unique* C^2-*solution to the* $U-$*ODE:*

$$\mathbb{E}\left[\int_0^\infty e^{-rt}(h(p_t) - g(\beta q(p_t)))\right],\ p_0 = p$$

- $U'(p) - q(p)$ *has an analytic solution. Moreover, when* $\kappa = 0$*, if*

$$\frac{M-m}{\psi} \leq \frac{\sqrt{2r\sigma_\xi^2}(r\sigma_\xi + \sigma_\theta)^2}{4\sigma_\theta^2},$$

$a^*(\cdot) = \beta q(\cdot)/\psi$ *is a* ***Markov equilibrium***

Literature

- **Symmetric learning in reputation games (no experimentation)**
  - Holmstrom (1999), Board and Meyer-ter-Vehn (2013, 2014), Kovrijnykh (2007), Martinez (2009), Bar-Isaac and Deb (2014)
- **Symmetric learning and experimentation (linear payoffs and poisson learning)**
  - Bergeman and Hege (2005), Horner and Samuelson (2014), Bonatti and Hörner (2014)
- **First-order approach to contracting**
  - Williams (2011), Sannikov (2015), Prat and Jovanovic (2014), DeMarzo and Sannikov (2015)
- **Ratchet effect**
  - Weitzman (1980), Laffont and Tirole (1988), Martinez (2009), Bhaskar (2014)

Conclusions

- Class of games of symmetric learning and imperfect monitoring
  - Necessary and sufficient conditions for Markov equilibrium
  - $\exists$ Markov eqbm. and simple tool for computation
  - Generality is important: (i) uncovers new economic insights; (ii) expands class of applications
- Regarding the key assumptions of the model:
  - **N long-run players**
  - Beyond **symmetric uncertainty**: Bonatti, Cisternas and Toikka (2017)
  - **Experimentation** and first-order approach

Technical Conditions

Assumption

(i) Differentiability: $u \in C^1(\mathbb{R})$, $\chi \in C^1(\mathbb{R} \times A)$ *and* $g \in C^2(A; \mathbb{R}_+)$ *with*

$$\rho := (g')^{-1} \in C^2(\mathbb{R}).$$

(ii) Growth conditions: the partial derivatives χ_p *and* χ_{a^*} *are bounded in* $\mathbb{R} \times A$, *and* u, u' *and* g' *have polynomial growth.*

(iii) Strong convexity: $g''(\cdot) \geq \psi$ *for some* $\psi > 0$.

▸ Back to the model

Dealing with Off-Path Private Beliefs

- Market constructs beliefs using q, and (*) holds, there is $\Gamma > 0$ s.t.

$$U(p^*) + \underbrace{[U'(p^*) - q(p^*)]}_{\textbf{info rent}}(p - p^*) + \Gamma \frac{(p - p^*)^2}{2}$$

  is an upper bound to the LRp's payoff

- In particular, $U(\cdot)$ is an upper bound when $p = p^*$
- By construction, $U(\cdot)$ is attained $q \Rightarrow q$ is optimal
- "Tight upper bounds"
  - Principle behind HJB equations
  - Williams (2011); Prat and Jovanovic (2014), Sannikov (2015)

▸ Verification Theorem