Online Appendix to "Signaling with Private Monitoring" (not for publication)

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S.1 Section 3: Omitted Proofs

Lemma S.1. *The process L is the belief about θ held by an outsider who observes only X. Moreover,* $\begin{pmatrix} \theta \\ \hat{M}_t \end{pmatrix} | \mathcal{F}_t^X \sim \mathcal{N}(M_t^{out}, \gamma_t^{out})$ *where* $M_t^{out} = \begin{pmatrix} L_t \\ L_t \end{pmatrix}$ *and* $\gamma_t^{out} = \begin{pmatrix} \frac{\gamma_{1t}}{1-\chi_t} & \frac{\gamma_{1t}\chi_t}{1-\chi_t} \\ \frac{\gamma_{1t}\chi_t}{1-\chi_t} & \frac{\gamma_{1t}\chi_t}{1-\chi_t} \end{pmatrix}$.

Proof. The outsider jointly filters the state $v_t = (\theta, \hat{M}_t)'$. For the evolution of the state and the signal, we adopt notation from Section 12.3 in Liptser and Shiryaev (1977). From the outsider's perspective, both players are on the equilibrium path, and thus the outsider believes that v_t evolves as

$$dv_t = a_1(t, X^{out})v_t dt + b_1(t, X)dW_1(t) + b_2(t, X)dW_2(t),$$

where $a_1(t, X^{out}) := \begin{pmatrix} 0 & 0 \\ \alpha_{3t}^2\gamma_{1t} & -\alpha_{3t}^2\gamma_{1t} \end{pmatrix}$, $b_1(t, X) := \begin{pmatrix} 0 & 0 \\ 0 & \frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y} \end{pmatrix}$, $b_2(t, X) := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $W_1(t) := \begin{pmatrix} W_{11}(t) \\ Z_t^Y \end{pmatrix}$ and $W_2(t) := Z_t^X$, where $W_{11}(t)$ is a standard Brownian motion and $W_{11}(t), Z_t^Y$ and Z_t^X are mutually independent. The signal is

$$dX_t^{out} := dX_t - [\delta_{0t} + \delta_{2t}L_t]dt = A_1(t, X)v_t + B_1(t, X)W_1(t) + B_2(t, X)W_2(t),$$

where $A_1(t, X) := \begin{pmatrix} 0 & \delta_{1t} \end{pmatrix}$, $B_1(t, X) := \begin{pmatrix} 0 & 0 \end{pmatrix}$ and $B_2(t, X) = \sigma_X$.

Hence, denoting $M_t^{out} = \begin{pmatrix} M_{t,1}^{out} \\ M_{t,2}^{out} \end{pmatrix}$ and $\gamma_t^{out} = \begin{pmatrix} \gamma_{t,11}^{out} & \gamma_{t,12}^{out} \\ \gamma_{t,21}^{out} & \gamma_{t,22}^{out} \end{pmatrix}$ and imposing $\gamma_{t,21}^{out} = \gamma_{t,12}^{out}$, we have from the standard filtering equations of Liptser and Shiryaev (1977, Theorem 12.7) that $\begin{pmatrix}\theta \\ \hat{M}_t\end{pmatrix} |\mathcal{F}_t^X \sim \mathcal{N}(M_t^{out}, \gamma_t^{out})$, where M^{out} and γ^{out} are the unique solutions to

$$dM_t^{out} = a_1(t,X)M_t^{out} + \frac{1}{\sigma_X^2}\left[\begin{pmatrix}0\\0\end{pmatrix} + \gamma_t^{out}\begin{pmatrix}0\\ \delta_{1t}\end{pmatrix}\right] \times \left\{dX_t^{out} - (\delta_{1t}M_{t,2}^{out})dt\right\} \tag{S.1}$$
$$\dot{\gamma}_t^{out} = a_1(t,X)\gamma_t^{out} + \gamma_t^{out}a_1^* + \frac{1}{\sigma_X^2}\left[\begin{pmatrix}0\\0\end{pmatrix} + \gamma_t^{out}\begin{pmatrix}0\\ \delta_{1t}\end{pmatrix}\right]\left[\begin{pmatrix}0\\0\end{pmatrix} + \gamma_t^{out}\begin{pmatrix}0\\ \delta_{1t}\end{pmatrix}\right]^*$$

with initial conditions $M_0^{out} = \begin{pmatrix}\mu\\ \mu\end{pmatrix}$ and $\gamma_0^{out} = \begin{pmatrix}\gamma^o & 0\\ 0 & 0\end{pmatrix}$.

Recall that γ_1 and χ satisfy $\dot{\gamma}_{1t} = -\frac{\alpha_{3t}^2}{\sigma_Y^2}\gamma_{1t}^2$ and $\dot{\chi}_t = \frac{\gamma_{1t}\alpha_{3t}^2(1-\chi_t)}{\sigma_Y^2} - \frac{\gamma_{1t}\delta_{1t}^2\chi_t^2}{\sigma_X^2}$ with initial conditions $\gamma_{10} = \gamma^o$ and $\chi_0 = 0$. It is straightforward to verify that $\gamma_t^{out} = \begin{pmatrix}\frac{\gamma_{1t}}{1-\chi_t} & \frac{\gamma_{1t}\chi_t}{1-\chi_t}\\ \frac{\gamma_{1t}\chi_t}{1-\chi_t} & \frac{\gamma_{1t}\chi_t}{1-\chi_t}\end{pmatrix}$ satisfies the γ^{out}-ODE above along with the given initial condition. Moreover, γ_t^{out} is positive semidefinite as its leading principal minors are positive multiples of 1 and $\chi - \chi^2 > 0$.

Next, substitute given the solution γ_t^{out} into (S.1) and subtract the equation for the second component from its first to obtain the following SDE for $\bar{M} := M_1^{out} - M_2^{out}$: $d\bar{M}_t = -\bar{M}_t\alpha_{3t}^2\gamma_{1t}/\sigma_Y^2$ with initial condition $\bar{M}_0 = 0$. Now if $\bar{M}_t > 0$, then $d\bar{M}_t < 0$, giving us a contradiction; likewise for the case $\bar{M}_t < 0$. It follows that $\bar{M}_t = 0$, and thus $M_{t,1}^{out} = M_{t,2}^{out}$, for all $t \geq 0$. Substituting this back into (S.1), we have

$$dM_{t,1}^{out} = \frac{\gamma_{1t}\delta_{1t}\chi_t}{\sigma_X^2(1-\chi_t)}(dX_t^{out} - \delta_{1t}M_{t,1}^{out}dt) = \frac{\gamma_{1t}\delta_{1t}\chi_t}{\sigma_X^2(1-\chi_t)}[dX_t - (\delta_{0t} + M_{1,t}^{out}\delta_{1t} + L_t\delta_{2t})dt].$$

On the other hand, we have

$$dL_t = \frac{L_t[\hat{\mu}_{1t} + \hat{\mu}_{2t} + \hat{\mu}_{3t}]dt + \hat{\mu}_{0t}dt + \hat{B}_t dX_t}{1-\chi_t} = \frac{\gamma_{1t}\delta_{1t}\chi_t\,[dX_t - (\delta_{0t} + L_t(\delta_{1t} + \delta_{2t}))dt]}{\sigma_X^2(1-\chi_t)}.$$

Hence $\bar{L}_t := M_{t,1}^{out} - L_t$ satisfies $d\bar{L}_t = -\frac{\gamma_{1t}\delta_{1t}\chi_t}{\sigma_X^2(1-\chi_t)}\bar{L}_t\delta_{1t}$ with initial condition $\bar{L}_0 = \mu - \mu = 0$. We conclude that $\bar{L}_t = 0$, and thus $L_t = M_{t,1}^{out} = M_{t,2}^{out}$, for all $t \geq 0$. □

S.2 Section 4: Omitted Proofs

S.2.1 Proof of Proposition 1

As described in Appendix B, we obtain a system of ODEs for $(v_0, v_1, v_3, \beta_0, \beta_1, \beta_3)$: $\dot{v}_{0t} = rv_{0t} + \frac{\beta_{0t}^2}{4} + \frac{\beta_{3t}}{4}\gamma_t(\beta_{3t} - \beta_{1t})$, $\dot{v}_{1t} = rv_{1t} - \frac{\beta_{0t}}{2}$, and $\dot{v}_{3t} = \frac{1}{4} + rv_{3t} - \frac{\beta_{3t}^2}{2}$ along with the ODEs for $(\beta_0, \beta_1, \beta_3)$ shown in that section, with terminal conditions $(v_{0T}, v_{1T}, v_{3T}, \beta_{0T}, \beta_{1T}, \beta_{3T}) = (0, 0, 0, 0, 1/2, 1/2)$. Solving the subsystem $(\beta_0, \beta_1, \beta_3, \gamma)$ delivers the remaining v_i, as their ODEs are uncoupled from one another and linear in themselves.

We now solve the BVP by transforming it into a backward IVP: abusing notation,

$$(\dot\beta_{0t}, \dot\beta_{1t}, \dot\beta_{3t}, \dot\gamma_t) = \beta_{3t} \times \left(-2r\beta_{0t}, r(1-2\beta_{1t}) - \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2}, r(1-2\beta_{3t}) + \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y}, \frac{\beta_{3t}\gamma_t^2}{\sigma_Y^2}\right), \quad \text{(S.2)}$$

with initial conditions $\beta_{00} = 0$, $\beta_{10} = \beta_{30} = \frac{1}{2}$ and $\gamma_0 = \gamma^F \geq 0$. Define $B_t^{Pub} := \beta_{1t} + \beta_{3t}$. We shall argue via the intermediate value theorem that there exists γ^F such that $\gamma_T = \gamma^o$, thus solving the BVP. To that end, we make use of the following lemma, which establishes uniform bounds and other key properties for the equilibrium coefficients.

Lemma S.2. *Fix any $\gamma^F \geq 0$. If a solution to the backward system exists over $[0, T]$, then any such solution must have the following properties. If $\gamma^F > 0$, then (i) $B_t^{Pub} = 1$ for all $t \in [0, T]$, (ii) $\beta_{3t} \in (1/2, 1)$ and $\beta_{1t} \in (0, 1/2)$ for all $t \in (0, T]$, (iii) β_3 is monotonically increasing while β_1 is monotonically decreasing, and (iv) γ is strictly increasing. If $\gamma^F = 0$, then $\beta_{1t} = \beta_{3t} = \frac{1}{2}$ and $\gamma_t = 0$ for all $t \in [0, T]$. For any $\gamma^F \geq 0$, $\beta_0 \equiv 0$.*

Proof of Lemma S.2. Because the system (S.2) is C^1, the solution is unique when it exists. If $\gamma^F = 0$, it is clear by inspection that $(\beta_0, \beta_1, \beta_3, \gamma) = (0, 1/2, 1/2, 0)$ (uniquely) solves the IVP, so assume hereafter that $\gamma^F > 0$. We first claim that $\beta_3 > 0$. Indeed, let $f^{\beta_3}(t, \beta_{3t})$ denote the RHS of the β_3-ODE in (S.2). Letting $x_t := 0$ for all $t \in [0, T]$, we have $\beta_{30} = 1/2 > x_0$ and $\dot\beta_{3t} - f^{\beta_3}(t, \beta_{3t}) = 0 = \dot x_t - f^{\beta_3}(t, x_t)$; by the comparison theorem, the claim follows. Now, add the ODEs that β_1 and β_3 satisfy to get $\dot B_t^{Pub} = 2r\beta_{3t}(1 - B_t^{Pub})$ with $B_0^{Pub} = 1$; because the RHS is of class C^1, it has a unique solution, which is clearly $B^{Pub} = 1$. Hence, $\beta_1 + \beta_3 = 1$ and $\dot\beta_{3t} = \beta_{3t}\left[r(1-2\beta_{3t}) + \frac{\beta_{3t}(1-\beta_{3t})\gamma_t}{\sigma_Y^2}\right]$, and we maintain the label $f^{\beta_3}(t, \beta_{3t})$ for its RHS. Defining $x_t := 1$ for all $t \in [0, T]$, then, $x_0 = 1 > \beta_{30} = \frac{1}{2}$, and $\dot\beta_{3t} - f^{\beta_3}(t, \beta_{3t}) = 0 \leq r = \dot x_t - f^{\beta_3}(t, x_t)$; thus, $\beta_3 < 1$ and $\beta_1 = 1 - \beta_3 > 0$.

Since $\beta_3 > 0$, γ is clearly strictly increasing, and hence $\gamma_t > 0$ for all $t \in [0, T]$. Now, $\dot\beta_{3t} = \frac{1}{2}\left[0 + \frac{\gamma_t}{4\sigma_Y^2}\right] > 0$ whenever $\beta_{3t} = \frac{1}{2}$, and thus $\beta_{3t} > 1/2$ and $\beta_{1t} < 1/2$ for all $t \in (0, T]$.

We now turn to (iii). Since $\dot\beta_{1t} + \dot\beta_{3t} = 0$, we just show that $\dot\beta_3 > 0$; in turn, it suffices to show that $H_t := \dot\beta_{3t}/\beta_{3t} = r(1-2\beta_{3t}) + \frac{\beta_{3t}(1-\beta_{3t})\gamma_t}{\sigma_Y^2} > 0$ for all $t \in [0, T]$. Observe that $H_0 = \frac{\gamma_0}{4\sigma_Y^2} > 0$, and with algebra it can be shown that if $H_t = 0$, $\dot H_t = \frac{(1-\beta_{3t})\beta_{3t}^3\gamma_t^2}{\sigma_Y^4} > 0$. It follows that $H > 0$ as desired. Finally, note that in all cases, we have $\beta_3 > 0$, so from (S.2) $\beta_0 \equiv 0$. Also, as long as $\gamma^F > 0$, $\gamma > 0$, so (v_2, v_4, v_5) are well defined. □

Given the uniform bounds established in Lemma S.2, we solve the BVP through a *shooting* step, arguing by contradiction as in Bonatti et al. (2017). Note that if $\gamma^F = 0$, the IVP has the (unique) static solution. Define

$$\bar{\gamma} := \sup\{\tilde{\gamma}^F > 0 \mid \text{a solution to the IVP exists over } [0,T] \text{ for all } \gamma^F \in (0,\tilde{\gamma}^F)\}.$$

Since the right-hand side of the equations that comprise the IVP are of class C^1, the solution is unique when it exists, and there is continuous dependence of the solution on the initial conditions; in particular, the terminal value γ_T is continuous in γ^F (see Theorem on page 397 in Hirsch et al. (2004)). Hence if there exists $\gamma^F \in (0,\bar{\gamma})$ such that $\gamma_T(\gamma^F) \geq \gamma^o$, by the intermediate value theorem there exists a $\gamma^F \in (0,\bar{\gamma})$ such that $\gamma_T(\gamma^F) = \gamma^o$, allowing us to construct a solution to the BVP.

Suppose then that for all $\gamma^F \in (0,\bar{\gamma})$, $\gamma_T(\gamma^F) < \gamma^o$. In particular, because γ_t is non-decreasing in the backward system for any initial condition, we have that $\gamma_t \in (0,\gamma^o)$ does not explode and the uniform bounds from the lemma apply. We first claim that a solution to the IVP for $\gamma^F = \bar{\gamma}$ must exist over $[0,T]$. To see this, let $[0,\tilde{T})$ denote the maximal interval of existence, and suppose by way of contradiction that $\tilde{T} \in (0,T]$. Thus, there must be some function $x(\cdot,\bar{\gamma})$ which explodes at $\tilde{T}$, and so, for $\tilde{t} \in (0,\tilde{T})$ sufficiently close to $\tilde{T}$, we have $x(\tilde{t},\bar{\gamma}) \notin [0,1]$. But for any sequence $(\gamma_n^F)_{n\in\mathbb{N}}$ taking values in $(0,\bar{\gamma})$ such that $\gamma_n^F \uparrow \bar{\gamma}$, by continuity of solutions with respect to initial conditions, we have $x(\tilde{t},\bar{\gamma}) = \lim_{n\to\infty} x(\tilde{t},\gamma_n^F) \in [0,1]$, a contradiction. We conclude that a solution to the IVP for $\gamma^F = \bar{\gamma}$ must exist over $[0,T]$, and hence, by the extensibility of the solutions (Theorem on page 397 in Hirsch et al. (2004)), that a solution must also exist for all $\gamma^F \in [\bar{\gamma},\bar{\gamma}+\epsilon)$, some $\epsilon > 0$, thereby violating the definition of $\bar{\gamma}$ as a supremum.

Thus, a solution to the BVP exists. Moreover, the lemma establishes (for reversed time) the properties of the coefficients stated in Proposition 1. As $\beta_3 \geq 1/2$ is finite, we have $\beta_{3t}, \gamma_t > 0$ allowing us to recover (v_2, v_4, v_5) through the identities given in Appendix B, and then (v_0, v_1, v_3) are pinned down as argued above, and thus we have characterized an LME with the stated properties.

S.2.2 Proof of Proposition 3

We begin by deriving a representation for the leader's second-order belief.

Lemma S.3 (Belief Representation)**.** *Suppose that the follower expects $a_t = \alpha_{2t}\mu + \alpha_t\theta$, where $\alpha_2 = \beta_{2t} + \beta_{1t}(1-\chi_t)$, $\alpha = \beta_3 + \beta_1\chi$, $\chi = 1 - \gamma/\gamma^o$, and $\gamma_t := \hat{\mathbb{E}}_t[(\theta - \hat{M}_t)^2]$. Then $\dot{\gamma}_t = -\left(\frac{\gamma_t\alpha_t}{\sigma_Y^2}\right)^2$. Moreover, if the leader follows $a_t = \alpha_{2t}\mu + \alpha_t\theta$, $M_t = \chi_t\theta + (1-\chi_t)\mu$ holds at all times.*

Proof of Lemma S.3. Anticipating $a_t = \alpha_{2t}\mu + \alpha_t\theta$, the receiver's belief is $\sim \mathcal{N}(M_t, \gamma_t)$ where

$d\hat{M}_t = \frac{\alpha_t\gamma_t}{\sigma_Y^2}[dY_t - (\alpha_{2t}\mu + \alpha_t\hat{M}_t)dt]$ and $\dot{\gamma}_t = -\frac{\gamma_t^2\alpha_t^2}{\sigma_Y^2}$. Thus, $\hat{M}_t = \mu R(t,0) + \int_0^t R(t,s)\frac{\alpha_s\gamma_s}{\sigma_Y^2}[(a_s - \alpha_{2s}\mu)ds + \sigma_Y dZ_s^Y]$ and $M_t = \mu R(t,0) + \int_0^t R(t,s)\frac{\alpha_s\gamma_s}{\sigma_Y^2}(a_s - \alpha_{2s}\mu)ds$ where $R(t,s) = \exp(-\int_s^t \frac{\alpha_u^2\gamma_u}{\sigma_Y^2}du)$. Solving for M after inserting $a_t = \beta_{1t}M_t + \beta_{2t}\mu + \beta_{3t}\theta$, and imposing the representation, it is easy to conclude that $M_t = \chi_t\theta + (1-\chi_t)\mu$ will hold if and only if $\dot{\chi}_t = \frac{\alpha_t^2\gamma_t}{\sigma_Y^2}(1-\chi_t)$. By arguments analogous to those used for Lemma 3, the (γ,χ)-ODE pair admits a unique solution, and it satisfies $\chi = 1 - \gamma/\gamma^o$. □

If the leader uses $a_t = \beta_{1t}M_t + \beta_{2t}\mu + \beta_{3t}\theta$ then, using the representation $M_t = \chi_t\theta + (1-\chi_t)\mu$, $\hat{a}_t = \hat{\mathbb{E}}_t[a_t] = \alpha_{2t}\mu + \alpha_t\hat{M}_t$. Taking an expectation in the leader's flow payoff $\frac{1}{4}[-(a_t-\theta)^2 - (a_t - \hat{a}_t)^2]$ then yields that (θ, M_t, t) is the relevant state on and off path. (Indeed, expanding the squares in the previous expression the only nontrivial component is $\mathbb{E}_t[\hat{a}_t^2]$, which makes $\mathbb{E}_t[\hat{M}_t^2]$ appear; however, $\mathbb{E}_t[\hat{M}_t^2] = M_t^2 + \mathbb{E}_t[(\hat{M}_t - M_t)^2] = M_t^2 + \gamma_t\chi_t$ after all private histories.[1])

We can then set up the HJB equation. Since $dM_t = \frac{\alpha_t\gamma_t}{\sigma_Y^2}(a - \alpha_{2t}\mu - \alpha_t m)\,dt$ from the proof of Lemma S.3,

$$rV = \sup_{a\in\mathbb{R}}\left\{\frac{1}{4}[-(a-\theta)^2 - \left(a^2 - 2a[\alpha_{2t}\mu + \alpha_t m] + \alpha_{2t}^2\mu^2 + 2\alpha_{2t}\alpha_t\mu m + \alpha_t^2[m^2 + \gamma_t\chi_t]\right)] \right.$$
$$\left. + V_t + \frac{\alpha_t\gamma_t}{\sigma_Y^2}(a - \alpha_{2t}\mu - \alpha_t m)V_m\right\}.$$

We then guess $V(\theta, m, \mu, t) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\mu + v_{4t}\theta^2 + v_{5t}m^2 + v_{6t}\mu^2 + v_{7t}\theta m + v_{8t}\theta\mu + v_{9t}m\mu$ and take analogous steps to those in the proof of Proposition 1.

We first note that there is a core BVP consisting of $(\beta_1, \beta_2, \beta_3, \gamma)$. (As in the proof of Proposition 2, $\beta_0 = 0$.) Second, we construct a backward IVP version of our original BVP that has a parametrized initial condition γ^F for the $\gamma-$ODE:

$$\dot{\beta}_{1t} = \alpha_t(2\sigma_Y^2)^{-1}\times\left\{r\sigma_Y^2 - 2\beta_{1t}[\beta_{3t}\gamma_t + r\sigma_Y^2(2-\chi_t)] + 2\beta_{1t}^2\gamma_t(1-\chi_t)\right\} \quad \text{(S.3)}$$
$$\dot{\beta}_{2t} = \alpha_t(2\sigma_Y^2)^{-1}\times\left\{-2r\sigma_Y^2\beta_{2t}(2-\chi_t) + r\sigma_Y^2(1-\chi_t) - 2\gamma_t\beta_{1t}^2(1-\chi_t)\right\} \quad \text{(S.4)}$$
$$\dot{\beta}_{3t} = \alpha_t(2\sigma_Y^2)^{-1}\times\left\{r\sigma_Y^2(2-\chi_t) + 2\beta_{3t}[\beta_{1t}\gamma_t - r\sigma_Y^2(2-\chi_t)]\right\} \quad \text{(S.5)}$$
$$\dot{\gamma}_t = \alpha_t^2\gamma_t^2/\sigma_Y^2 \quad \text{(S.6)}$$

with initial condition $(\beta_{10}, \beta_{20}, \beta_{30}, \gamma_0) = (\frac{1}{2(2-\chi_0)}, \frac{1-\chi_0}{2(2-\chi_0)}, \frac{1}{2}, \gamma^F)$ and where $\chi = 1 - \gamma/\gamma^o$.

We aim to prove that there exists $\gamma^F \in (0, \gamma^o)$ such that the IVP has a (unique) solution which satisfies $\gamma_T = \gamma^o$. ($\gamma^F = 0$ cannot work, as $(\beta_1, \beta_2, \beta_3, \gamma) = (1/2, 0, 1/2, 0)$ is the

[1]From the proof of Lemma S.3, $\mathbb{E}_t[(\hat{M}_t - M_t)^2] = \mathbb{E}_t[(\int_0^t R(t,s)\frac{\alpha_s\gamma_s}{\sigma_Y}dZ_s^Y)^2] = \int_0^t R(t,s)^2\frac{\alpha_s^2\gamma_s^2}{\sigma_Y^2}ds = \int_0^t \exp(2\int_s^t \frac{\dot{\gamma}_u}{\gamma_u}du)(-\dot{\gamma}_s)ds = \int_0^t (\gamma_t/\gamma_s)^2(-\dot{\gamma}_s)ds = \gamma_t^2(1/\gamma_t - 1/\gamma^o) = \gamma_t\chi_t$.

unique solution.) As argued in the proof of Proposition 1, it suffices to show that the system is uniformly bounded if $\gamma_t \in [0, \gamma^o]$ over $[0, T]$.

The α-ODE is $\dot{\alpha}_t = f^\alpha(t, \alpha_t) := r\alpha_t[1 - \alpha_t(2 - \chi_t)]$ and $\alpha_0 = \frac{1}{2-\chi_0} > 0$. By the comparison theorem, $\alpha > 0$; hence, by the same argument as in the proof of Lemma S.2, γ is increasing (in the backward system), so $\chi = 1 - \gamma/\gamma^o < 1$ is decreasing. As $\alpha_0 = \frac{1}{2-\chi_0}$ and $\dot{\alpha}_0 > \frac{d}{dt}\left(\frac{1}{2-\chi_t}\right)|_{t=0}$, the comparison theorem can be applied to α and $1/(2-\chi)$ to show $\alpha_t \geq 1/(2-\chi_t) \geq 1/2$, with both inequalities strict for all $t \in (0, T]$, for all $r \geq 0$; in turn, $\dot{\alpha}_t \leq 0$ (and hence $\dot{\alpha}_t \geq 0$ in the forward system) for all $t \in [0, T]$, with strict inequality for $t \in (0, T]$ if and only if $r > 0$. It follows that for all $t \in (0, T]$, $\alpha_t \leq \alpha_0 = \frac{1}{2-\chi_0} < 1$.

Now, $B^{NF} := \beta_1 + \beta_2 + \beta_3$ satisfies $\dot{B}^{NF}_t = \frac{\alpha_t}{2\sigma_Y^2}\left\{2r\sigma_Y^2(2-\chi_t)[1 - B^{NF}_t]\right\}$ with $B^{NF}_0 = 1$; thus $B^{NF} \equiv 1$. This establishes that in any LME, $a_t = (1 - \alpha_{3t})\mu + \alpha_{3t}\theta$. By routine application of the comparison theorem to the backward system, $\beta_3 \in (1/2, 1)$ and $\beta_1 \in (0, 1)$, from which β_2 is bounded too; thus, a solution to the BVP for $(\beta_1, \beta_2, \beta_3, \gamma)$ exists, as discussed in the proof of Proposition 1. Now $\alpha > 0$ and $\gamma > 0$ since α is finite, so from $(\beta_0, \beta_1, \beta_2, \beta_3)$, the coefficients (v_2, v_5, v_7, v_9) are backed out directly as in the proof of Theorem 1. The ODEs for the remaining value function coefficients are linear and uncoupled, so they also have unique solutions.

The final claim is $\alpha_T \to 1$ as $T \to \infty$ in the forward system. Indeed, since $\alpha > 1/2$, we have $\gamma_T \to 0$ as $T \to \infty$; thus $\chi_T \to 1$ and $\alpha_T = 1/(2 - \chi_T) \to 1$, all in forward form.

S.2.3 Proof of Proposition 4

We begin with two lemmas that provide closed-form solutions for the public and no feedback cases when $r = 0$.

Lemma S.4 (Closed-form solution public case $r = 0$)**.** *For $r = 0$, the coordination game has a unique LME for the public case, and $(\beta_0, \beta_1, \beta_3, \gamma)$ satisfy $\beta_0 \equiv 0$, $\beta_1 \equiv 1 - \beta_3$,*

$$\gamma_t = \frac{\gamma_T}{2} + \frac{1}{\frac{2}{\gamma_T} - \frac{T-t}{\sigma_Y^2}}, \beta_{3t} = \frac{1}{2 - \frac{\gamma_T(T-t)}{2\sigma_Y^2}}, \text{ and } \gamma_T = \frac{\gamma^o T + 2\sigma_Y^2 - \sqrt{(\gamma^o T)^2 + 4\sigma_Y^4}}{T}. \quad \text{(S.7)}$$

Proof. Observe that $\dot{\beta}_{3t}\gamma_t + \beta_{3t}\dot{\gamma}_t = \frac{\beta_{3t}^2\gamma_t^2}{\sigma_Y^2}$. Hence, define $\Pi_t := \beta_{3t}\gamma_t$, which has ODE $\dot{\Pi}_t = \frac{\Pi_t^2}{\sigma_Y^2}$ with initial condition $\Pi_0 = \beta_{30}\gamma^F = \gamma^F/2$; its solution is $\Pi_t = \left[\frac{2}{\gamma^F_{Pub}} - \frac{t}{\sigma_Y^2}\right]^{-1}$. Substitute Π into $\dot{\gamma}_t = -\frac{\beta_{3t}^2\gamma_t^2}{\sigma_Y^2}$ to obtain $\dot{\gamma}_t = \frac{1}{\sigma_Y^2}\left[\frac{2}{\gamma^F_{Pub}} - \frac{t}{\sigma_Y^2}\right]^{-2}$ which implies $\gamma_t = C_\gamma + \left[\frac{2}{\gamma^F_{Pub}} - \frac{t}{\sigma_Y^2}\right]^{-1}$. As

$\gamma_0 = \gamma^F_{Pub}$, we have $C_\gamma = \gamma^F_{Pub}/2$ and thus

$$\gamma_t = \frac{\gamma^F_{Pub}}{2} + \left[\frac{2}{\gamma^F_{Pub}} - \frac{t}{\sigma^2_Y}\right]^{-1}. \tag{S.8}$$

Moreover, $\gamma_T = \gamma^o = \frac{\gamma^F_{Pub}}{2} + \left[\frac{2}{\gamma^F_{Pub}} - \frac{T}{\sigma^2_Y}\right]^{-1}$, which is equivalent to the quadratic $\frac{T}{2}\left(\gamma^F_{Pub}\right)^2 - \left(\gamma^o T + 2\sigma^2_Y\right)\gamma^F_{Pub} + 2\sigma^2_Y\gamma^o = 0$. The quadratic on the LHS is convex and evaluates to $2\sigma^2_Y\gamma^o > 0$ at $\gamma^F_{Pub} = 0$ and evaluates to $-\left(\gamma^o\right)^2 T/2 < 0$ at $\gamma^F_{Pub} = \gamma^o$, so there is a unique solution in $(0, \gamma^o)$ which in the forward system is γ_T as in the proposition statement. Substituting this into (S.8) and returning to the forward system by replacing t with $T - t$ yields γ_t in the forward system. It is easy to verify that $\gamma_t > 0$ for all t.

Lastly, in the forward system, $\beta_{3t} = \Pi_t/\gamma_t = [2 - \frac{\gamma^{Pub}_T(T-t)}{2\sigma^2_Y}]^{-1}$ and $\beta_{1t} = 1 - \beta_{3t}$. □

Lemma S.5 (Closed-form solution no-feedback case $r = 0$)**.** *For $r = 0$, the coordination game has a unique LME for the no feedback case:*

$$\beta_{1t} = \frac{\gamma^o[(\gamma^o + \gamma_T)^2\sigma^2_Y - (T-t)(\gamma^o)^2\gamma_T]}{(\gamma^o + \gamma_T)[2\sigma^2_Y(\gamma^o + \gamma_T)^2 - (T-t)(\gamma^o)^2\gamma_T]}, \quad \beta_{3t} = \frac{\sigma^2_Y(\gamma^o + \gamma_T)^2}{2\sigma^2_Y(\gamma^o + \gamma_T)^2 - (T-t)(\gamma^o)^2\gamma_T}$$

$$\alpha_t = \frac{\gamma^o}{\gamma^o + \gamma_T}, \quad \gamma_t = \frac{\gamma_T\sigma^2_Y\left(\gamma^o + \gamma_T\right)^2}{\sigma^2_Y\left(\gamma^o + \gamma_T\right)^2 - (T-t)\left(\gamma^o\right)^2\gamma_T},$$

for all $t \in [0, T]$, where $\chi_t = 1 - \gamma_t/\gamma^o$ and $\gamma_T \in (0, \gamma^o)$ is the unique solution in $(0, \gamma^o)$ to the cubic $q(\gamma) := \gamma T\left(\gamma^o\right)^3 + \left(\gamma - \gamma^o\right)\left(\gamma + \gamma^o\right)^2\sigma^2_Y = 0$, and $\beta_0 \equiv 1 - \beta_1 - \beta_3$.

Proof. We work with the backward system, where the α-ODE is $\dot{\alpha}_t = r\alpha[1 - \alpha_t(2 - \chi_t)]$. With $r = 0$, α must be constant and equal to its initial value $\alpha_0 = \frac{1}{2-\chi_0}$. Next, recall that by Lemma S.3, $\chi_t = 1 - \frac{\gamma_t}{\gamma^o}$, so $\chi_0 = 1 - \frac{\gamma^F_{NF}}{\gamma^o}$ and thus $\alpha_t = \alpha = \frac{\gamma^o}{\gamma^F_{NF} + \gamma^o}$ for all $t \in [0, T]$. Next, note that the ODE $\dot{\gamma}_t = \frac{\alpha^2\gamma^2_t}{\sigma^2_Y}$ given an initial value γ^F_{NF} has solution $\gamma_t = \frac{\gamma^F_{NF}\sigma^2_Y}{\sigma^2_Y - \gamma^F_{NF}\left(\frac{\gamma^o}{\gamma^F_{NF}+\gamma^o}\right)^2 t}$; switching back to the forward system by replacing t with $T - t$ yields the expression in the original statement. Now the terminal condition $\gamma_T = \gamma^o$ is equivalent to the following cubic equation for γ^F_{NF}:

$$q(\gamma^F_{NF}) := \gamma^F_{NF}T\left(\gamma^o\right)^3 + \left(\gamma^F_{NF} - \gamma^o\right)\left(\gamma^F_{NF} + \gamma^o\right)^2\sigma^2_Y = 0. \tag{S.9}$$

Note $q\left(\gamma^F_{NF}\right) > 0$ for $\gamma^F_{NF} \geq \gamma^o$ and $q\left(\gamma^F_{NF}\right) \leq 0$ for $\gamma^F_{NF} \leq 0$, so all real roots must lie in $(0, \gamma^o)$. Now any root to the cubic must satisfy

$$\frac{T(\gamma^o)^3}{\gamma^o - \gamma^F_{NF}} = \sigma^2_Y\frac{(\gamma^F_{NF} + \gamma^o)^2}{\gamma^F_{NF}}. \tag{S.10}$$

The LHS of (S.10) is strictly increasing for $\gamma_{NF}^{F} \in (0, \gamma^o)$ while the RHS is strictly decreasing in this interval, so q has a unique real root. Returning to the β_1 ODE, using $\alpha = \beta_1\chi + \beta_3$, we have $\dot{\beta}_1 = \frac{\alpha\gamma_t\beta_{1t}}{\sigma_Y^2}(\alpha - \beta_{1t})$. This ODE can be solved by integration after moving $\beta_1(\alpha - \beta_1)$ to the LHS, and with algebra, one obtains (in the forward system) the expression in the proposition statement. One then obtains β_{3t} from these using $\beta_{3t} = \alpha - \beta_{1t}\chi_t$. □

Equipped with the previous two lemmas, we now prove the proposition. We begin with the claim $\gamma_T^{Pub} > \gamma_T^{NF}$. Recall that γ_T^{NF} is the unique positive root of the cubic equation $q(\gamma) = 0$ defined in Lemma S.5. At γ_T^{NF}, it is easy to deduce that q must cross 0 from below, and hence to prove the claim, it suffices to show that $q(\gamma_T^{Pub}) > 0$. By direct calculation,

$$q(\gamma_T^{Pub}) = +\frac{\sigma_Y^2}{T^3}\left(2\sigma_Y^2 - \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right)\left(2T\gamma^o + 2\sigma_Y^2 - \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right)^2 + (\gamma^o)^3\left(T\gamma^o + 2\sigma_Y^2 - \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right) = (\gamma^o)^4 T q_2(S), \qquad \text{where}$$

$$q_2(S) := 1 + 2S - \sqrt{1+4S^2} + S\left(2S - \sqrt{1+4S^2}\right)\left(2 + 2S - \sqrt{1+4S^2}\right)^2 \text{ and } S := \frac{\sigma_Y^2}{T\gamma^o}.$$

We now show that $q_2(S) > 0$ for all $S > 0$ (observe that $q_2(0) = 0$). Let $R(S) = 1 + 2S - \sqrt{1+4S^2}$; it is straightforward to verify that $R(0) = 0$ and that for all $S \geq 0$, $R'(S) > 0$ and $R(S) < 1$. Moreover, the inverse of R is the function $S : [0, 1) \to [0, \infty)$ characterized by $S(R) := \frac{R(2-R)}{4(1-R)}$. Hence, by change of variables, $q_2(S) > 0$ for all $S > 0$ iff $q_3(R) > 0$, where $q_3(R) := R - S(R)(1-R)(R+1)^2$. Now for $R \in [0, 1)$, $q_3(R) > 0$ if and only if $S(R) = \frac{R(2-R)}{4(1-R)} < \frac{R}{(1-R)(R+1)^2}$, if and only $q_4(R) := (2-R)(R+1)^2 < 4$. It is straightforward to verify that over the interval $[0, 1]$, $q_4(R)$ attains its maximum value of 4 at $R = 1$, and tracing our steps backwards this implies that $q(\gamma_T^{Pub}) > 0$.

Next, we prove the claim $\beta_{30}^{Pub} > \alpha_{30}^{NF}$. Using the associated expressions from Lemmas S.4 and S.5, this is equivalent to

$$\frac{1}{2 - \frac{\gamma_T^{Pub}T}{2\sigma_Y^2}} > \frac{\gamma^o}{\gamma^o + \gamma_T^{NF}} \iff \hat{\gamma} := \gamma^o\left(1 - \frac{\gamma_T^{Pub}T}{2\sigma_Y^2}\right) < \gamma_T^{NF}.$$

It suffices to show that $q(\hat{\gamma}) = T\hat{\gamma}(\gamma^o)^3 + (\hat{\gamma} - \gamma^o)(\hat{\gamma} + \gamma^o)^2\sigma_Y^2 < 0$. Using the expression for γ_T^{Pub} from Lemma S.4, one can show that

$$q(\hat{\gamma}) = -\frac{T(\gamma^o)^4}{2\sigma_Y^4}\left[(T\gamma^o)^2 + 2\sigma_Y^4 - T\gamma^o\sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right].$$

The expression in square brackets can be written as $\frac{x+y}{2} - \sqrt{xy} > 0$ where $x = (T\gamma^o)^2 > 0$

and $y = (T\gamma^o)^2 + 4\sigma_Y^4 > 0$, and thus $q(\hat{\gamma}) < 0$, concluding the proof that $\beta_{30}^{Pub} > \alpha^{NF}$.

We now turn to claim (i). We begin by calculating the leader's ex ante flow payoffs in both cases. To simplify expressions, we rescale payoffs to remove the scalar $\frac{1}{4}$.

Lemma S.6. *The leader's ex ante flow payoffs in the public and no-feedback cases are* $\gamma_t^{Pub}[(1-\beta_{3t})^2 + \beta_{3t}^2]$ *and* $(1-\alpha_t)^2\gamma^o + \alpha_t^2\gamma_t^{NF}$, *respectively.*

Proof. Ex ante flows are given $\hat{\mathbb{E}}_0\left[(\theta - a_t)^2 + (a_t - \hat{a}_t)^2\right]$, where $\hat{\mathbb{E}}_0$ is the leader's ex ante expectation operator. In the public case, $a_t = (1-\beta_{3t})M_t + \beta_{3t}\theta$ and $\hat{a}_t = M_t$, so we have

$$\hat{\mathbb{E}}_0\left[(\theta - a_t)^2 + (a_t - \hat{a}_t)^2\right] = \hat{\mathbb{E}}_0\left[(\theta - M_t)^2\left([1-\beta_{3t}]^2 + \beta_{3t}^2\right)\right] = \gamma_t^{Pub}[(1-\beta_{3t})^2 + \beta_{3t}^2],$$

where in the last step we have used the law of iterated expectations and the fact that the variance is deterministic: $\hat{\mathbb{E}}_0\left[(\theta - M_t)^2\right] = \hat{\mathbb{E}}_0\left[\hat{\mathbb{E}}_t[(\theta - M_t)^2]\right] = \gamma_t^{Pub}$.

In the no-feedback case, we have $a_t = (1-\alpha_t)\mu + \alpha_t\theta$ and $\hat{a}_t = (1-\alpha_t)\mu + \alpha_t\hat{M}_t$, so

$$\hat{\mathbb{E}}_0\left[(\theta - a_t)^2 + (a_t - \hat{a}_t)^2\right] = (1-\alpha_t)^2\hat{\mathbb{E}}_0\left[(\theta-\mu)^2\right] + \alpha_t^2\hat{\mathbb{E}}_0\left[(\theta - \hat{M}_t)^2\right] = (1-\alpha_t)^2\gamma^o + \alpha_t^2\gamma_t^{NF},$$

where we have used the definition of γ^o and $\hat{\mathbb{E}}_0\left[(\theta - \hat{M}_t)^2\right] = \hat{\mathbb{E}}_0\left[\hat{\mathbb{E}}_t[(\theta - \hat{M}_t)^2]\right] = \gamma_t^{NF}$. □

We now use the ex ante flow payoffs above to calculate and compare ex ante total payoffs for the leader. For $i \in \{Pub, NF\}$, let V^i denote the leader's ex ante expected payoff and define $\tilde{T}^i := \frac{T\gamma_T^i}{\sigma_Y^2}$. Define $\rho := \gamma_T^{NF}/\gamma^o$. We begin by proving that for $r = 0$, $V^{Pub} = -\sigma_Y^2\left\{\tilde{T}^{Pub}/2 - \ln\left[\frac{16-8\tilde{T}^{Pub}}{(4-\tilde{T}^{Pub})^2}\right]\right\}$ and $V^{NF} = -\sigma_Y^2\left\{\rho(1-\rho) - \ln\rho\right\}$.

First consider the public benchmark, where from Lemma S.6,

$$V^{Pub} = \mathbb{E}_0\left(\int_0^T\left[-(\theta - a_t)^2 - (a_t - \hat{a}_t)^2\right]dt\right) = -\int_0^T\left[\gamma_t\left([1-\beta_{3t}]^2 + \beta_{3t}^2\right)\right]dt.$$

Using the closed-form expressions for γ_t and β_{3t}, the integrand simplifies to

$$\gamma_t\left([1-\beta_{3t}]^2 + \beta_{3t}^2\right) = \frac{\gamma_T^{Pub}}{2}\left[1 + \frac{2t\gamma_T^{Pub}\sigma_Y^2}{(2\sigma_Y^2 - t\gamma_T^{Pub})(4\sigma_Y^2 - t\gamma_T^{Pub})}\right] = \frac{\gamma_T^{Pub}}{2}\left[1 + \frac{2\tilde{t}^{Pub}}{(2-\tilde{t}^{Pub})(4-\tilde{t}^{Pub})}\right],$$

where $\tilde{t}^{Pub} := t\gamma_T^{Pub}/\sigma_Y^2$. Using that the function $g : x \mapsto \frac{x}{(2-x)(4-x)}$ has antiderivative $\ln\left(\frac{(4-x)^2}{2-x}\right)$ and integrating the second term w.r.t. $\tilde{t}^{Pub}$ over $\left[0, \tilde{T}^{Pub}\right]$ and rescaling by $\sigma_Y^2/\gamma_T^{Pub}$, we obtain

$$V^{Pub} = -T\frac{\gamma_T^{Pub}}{2} - \sigma_Y^2\left(\ln\left[\frac{(4-\tilde{T}^{Pub})^2}{2-\tilde{T}^{Pub}}\right] - \ln 8\right) = -\sigma_Y^2\left\{\tilde{T}^{Pub}/2 - \ln\left[\frac{16-8\tilde{T}^{Pub}}{(4-\tilde{T}^{Pub})^2}\right]\right\}.$$

Next, consider the no feedback case, where by Lemma S.6,

$$V^{NF} = \mathbb{E}_0 \int_0^T \left[-(\theta - a_t)^2 - (a_t - \hat{a}_t)^2\right] dt = -\int_0^T (1-\alpha)^2 \gamma^o dt - \int_0^T \alpha^2 \gamma_t^{NF} dt. \quad \text{(S.11)}$$

Consider the first term on the RHS of (S.11), which reduces to $-T(1-\alpha)^2\gamma^o$. From Lemma S.5 we have $-T(1-\alpha)^2\gamma^o = -T\gamma^o \left(\frac{\rho}{1+\rho}\right)^2 = -\sigma_Y^2 \rho(1-\rho)$, where the last equality is simply a rearrangement of the equation $q(\gamma_T^{NF}) = 0$ using $\gamma_T^{NF} \equiv \rho\gamma^o$.

The second term on the RHS of (S.11) can be rewritten as

$$\int_0^T \sigma_Y^2 \frac{\dot{\gamma}_t^{NF}}{\gamma_t^{NF}} dt = \sigma_Y^2 (\ln \gamma_T - \ln \gamma_0) = \sigma_Y^2 \ln \frac{\gamma_T^{NF}}{\gamma^o} = \sigma_Y^2 \ln \rho,$$

and hence (S.11) is equivalent to $-\sigma_Y^2\{\rho(1-\rho) - \ln \rho\}$, as desired.

Next, we prove that $V^{NF} < V^{Pub}$ for all $T, \gamma^o, \sigma_Y^2 > 0$. Substituting in the expressions for γ_F^{Pub}, using $\tilde{T} := \frac{T\gamma^o}{\sigma_Y^2}$ and $\rho := \frac{\gamma_T^{NF}}{\gamma^o}$ and that the equation $q(\gamma_T^{NF}) = 0$ is equivalent to $\tilde{T} = \frac{(1-\rho)(1+\rho)^2}{\rho}$, and simplifying, we obtain

$$\begin{aligned} V^{NF} - V^{Pub} = \sigma_Y^2 \Bigg\{ & \rho(\rho - 1) + \ln \rho + 1 + \tilde{T}/2 - \frac{\sqrt{4 + \tilde{T}^2}}{2} \\ & - \ln\left[8\left(-\tilde{T} + \sqrt{4 + (\tilde{T})^2}\right)\right] + 2\ln\left[2 - \tilde{T} + \sqrt{4 + (\tilde{T})^2}\right]\Bigg\} = \frac{\sigma_Y^2}{2\rho} f(\rho), \end{aligned}$$

where $f(x) := A_1(x) + 2x \ln\left(\frac{A_2(x)^2}{A_3(x)}\right)$, for $A_1(x) := x^3 - 3x^2 + 3x + 1 - z(x), A_2(x) := x^3 + x^2 + x - 1 + z(x), A_3(x) := 8[x^3 + x^2 - x - 1 + z(x)]$ and $z(x) := \sqrt{4x^2 + (1-x)^2(1+x)^4}$.

We now show that $f(x) < 0$ for all $x \in (0,1)$, so that in particular $f(\rho) < 0$, from which the desired result follows. We begin by showing that $A_2(x) > 0$ and $A_3(x) > 0$ for all $x > 0$. By inspection, for all $x > 0$, we have $A_2(x) > A_3(x)/8$, and

$$A_3(x)/8 = (x-1)(x+1)^2 + \sqrt{4x^2 + (1-x)^2(1+x)^4} > (x+1)^2\left[x - 1 + \sqrt{(1-x)^2}\right] \geq 0.$$

Next, we apply the inequality $\ln(y) \leq 2\left(y^{\frac{1}{2}} - 1\right)$ for $y > 0$ using $y = \frac{A_2(x)^2}{A_3(x)} > 0$ to obtain

$$f(x) \leq A_1(x) + 4x\left(\frac{A_2(x)}{\sqrt{A_3(x)}} - 1\right). \quad \text{(S.12)}$$

For $x > 0$, the RHS of (S.12) is negative if and only if

$$A_2(x)/\sqrt{A_3(x)} < -A_1/(4x) + 1. \tag{S.13}$$

For $x \in (0, 1)$, the RHS of (S.13) is strictly positive:

$$\begin{aligned} -\frac{A_1}{4x} + 1 &= \frac{1}{4x}\left[\sqrt{4x^2 + (1-x)^2(1+x)^4} - x^3 + 3x^2 + x - 1\right] \\ &> \frac{1}{4x}\left[(1-x)(1+x)^2 - x^3 + 3x^2 + x - 1\right] = \frac{1}{2}[1 + x(1-x)] > 0. \end{aligned}$$

Hence, for $x \in (0, 1)$, (S.13) is equivalent to

$$0 > A_2(x)^2 - A_3(x)\left(-\frac{A_1}{4x} + 1\right)^2 = \frac{2}{x^2}\left[(1-x)^2 A_4(x) + A_5(x)z(x)\right] \tag{S.14}$$

where $A_4(x) = x^6 - 4x^4 - x^3 + 4x^2 + 3x + 1$ and $A_5(x) = x^5 - 3x^4 + x^3 + 2x^2 - 1$. Now by Descartes' rule of signs, A_5 has 3 sign changes and at most 3 positive real roots, counting multiplicity. It is easy to verify that there is a double root at $x = 1$, that $A_5(2) = -1 < 0$, and that $\lim_{x \to +\infty} A_5(x) = +\infty$, so there is a positive root at some $x > 2$. This implies there are no roots in $(0, 1)$. Since $A_5(0) = -1 < 0$, it follows that $A_5(x) < 0$ for all $x \in (0, 1)$. Thus, without signing $A_4(x)$, it suffices to show that $(1-x)^2|A_4(x)| < -A_5(x)z(x)$, or equivalently (since $A_5(x) < 0$)

$$0 > (1-x)^4 A_4(x)^2 - z(x)^2 A_5(x)^2 = 4(1-x)^4 x^5 (x^4 - 2x^2 - 3x - 1). \tag{S.15}$$

Now for $x \in (0, 1)$, we have in (S.15) $x^4 - 2x^2 - 3x - 1 < -2x^2 - 3x < 0$. Since the outside factor is positive, we have shown that $f(x) < 0$ for $x \in (0, 1)$, and thus $V^{NF} < V^{Pub}$. □

S.2.4 Proof of Proposition 5

To sign coefficients, we work with ODEs in backward form. Consider any LME. By the same argument as in the proof of Proposition 2, $\beta_0 \equiv 0$. Next, note that $(\gamma_0, \chi_0) \in (0, \gamma^o) \times (0, 1)$. As in the proof of Theorem 1, define $(\tilde{\beta}_2, \tilde{v}_6, \tilde{v}_8) := (\beta_2/(1-\chi), v_6\gamma/(1-\chi)^2, v_8\gamma/(1-\chi))$; also, define $\tilde{\beta}_3 := \beta_3 - 1$. Using the initial values $\beta_{10} = -\frac{\psi\gamma_0}{\sigma_Y^2 + \psi\gamma_0\chi_0} < 0$, and $\tilde{\beta}_{20} = \tilde{\beta}_{30} = \tilde{v}_{60} = \tilde{v}_{80} = 0$, it is tedious but straightforward to verify that $\dot{\tilde{\beta}}_{20} < 0$, $\dot{\tilde{\beta}}_{30} < 0$, $\dot{\tilde{v}}_{60} = 0 > \ddot{\tilde{v}}_{60}$, and $\dot{\tilde{v}}_{80} = 0 > \ddot{\tilde{v}}_{80}$. Hence, for all sufficiently small $t > 0$, for all $x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}$, we have $x_t < 0$. Define $\tau := \inf\{t \in (0, T] : x_t = 0 \text{ for some } x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}\}$ (and $\tau = \infty$ if this set is empty). Suppose by way of contradiction that $\tau \leq T$. By continuity, $x_\tau = 0$

for some x. We derive a contradiction by arguing via the comparison theorem that for all $t \in (0, \tau]$ and all $x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}$, $x_t < 0$. Write each ODE in the form $\dot{x}_t = f^x(x_t, t)$, and define $y \equiv 0$. Consider any $s \in (0, \tau)$; by the definition of τ, each variable is strictly negative over $(0, s]$, so in particular, for each x, we have $x_s < y_s = 0$. And by definition, $\dot{x}_t - f^x(x_t, t) = 0$ over $[0, T]$. Moreover, for $t \in [s, \tau]$,

$$\begin{aligned}
\dot{y}_t - f^{\beta_1}(y_t, t) &= -\frac{2\tilde{\beta}_2 \gamma_t \chi_t}{\sigma_X^2} \geq 0 \\
\dot{y}_t - f^{\tilde{\beta}_2}(y_t, t) &= \frac{\alpha_t \gamma_t (\sigma_X^2 \beta_{1t}^2 - 2\tilde{v}_{6t}\chi_t)}{\sigma_X^2 \sigma_Y^2} \geq 0 \\
\dot{y}_t - f^{\tilde{\beta}_3}(y_t, t) &= -\frac{\gamma_t (1 + \beta_{1t}\chi_t)[\sigma_X^2 \beta_{1t} + \tilde{v}_{8t}\chi_t]}{\sigma_X^2 \sigma_Y^2} \geq 0 \\
\dot{y}_t - f^{\tilde{v}_6}(y_t, t) &= \frac{1}{2}\tilde{\beta}_{2t}\gamma_t(2\beta_{1t} + \tilde{\beta}_{2t}) \geq 0 \\
\dot{y}_t - f^{\tilde{v}_8}(y_t, t) &= -\gamma_t[\tilde{\beta}_{2t} - \beta_{1t}\tilde{\beta}_{3t}] \geq 0,
\end{aligned}$$

where we have used that $\alpha > 0$ (since $\alpha_0 = \frac{\sigma_Y^2}{\sigma_Y^2 + \psi\gamma_0\chi_0} > 0$ and α does not change sign, as shown in the proof of Theorem 1), that for all $x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}$ and all $t \in [0, \tau]$, $x_t \leq 0$, and in the third line, that $1 + \beta_{1t}\chi_t \geq \beta_{3t} + \beta_{1t}\chi_t = \alpha_t > 0$. By the comparison theorem, we have $x_t < y_t = 0$ for all $t \in [s, \tau]$ and all $x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}$, contradicting that $x_\tau = 0$ for some such x. Hence, $\tau = \infty$, and we conclude that for all $t > 0$ going backward ($t < T$ going forward), $\beta_{1t} < 0$, $\beta_{2t} = \tilde{\beta}_{2t}(1 - \chi_t) < 0$, and $\beta_{3t} = \tilde{\beta}_{3t} + 1 < 1$, from which it follows that $\beta_{3t} = \alpha_t - \beta_{1t}\chi_t \geq \alpha_t > 0$. Moreover, since $\alpha_t = \beta_{3t} + \beta_{1t}\chi_t \leq \beta_{3t} < 1$ for $t \in [0, T)$. The remaining inequalities at time T (going forward) are immediate from the terminal conditions.

To show that, for small $r > 0$, α is has an interior minimum and is initially decreasing, we consider the system going forward in time. The α-ODE is

$$\dot{\alpha}_t = -r(1 - \alpha_t)\alpha_t - \frac{\alpha_t \gamma_t \chi_t}{\sigma_X^2 \sigma_Y^2} \left\{ \sigma_Y^2 \chi_t [2\tilde{\beta}_{2t} + \beta_{1t}] + \alpha_t \tilde{v}_{8t} \right\}, \tag{S.16}$$

and the terminal condition is $\alpha_T = \frac{\sigma_Y^2}{\sigma_Y^2 + \psi\gamma_T\chi_T} \in (0, 1)$. Given that $\chi_0 = 0$, (S.16) then yields $\dot{\alpha}_0 < 0$. In other words, α is initially decreasing.

To establish an interior minimum for small $r > 0$, suppose now by way of contradiction that there exists a sequence $(r_n)_{n \in \mathbb{N}}$ with $r_n \downarrow 0$ and an associated sequence of LME with coefficients $(\vec{\beta^n}, \vec{v^n}, \gamma^n, \chi^n)$ such that for all n, α^n does not have an interior minimum. As shown above, $\dot{\alpha}_0^n < 0$, so the minimum must occur at $t = T$, and $\dot{\alpha}_T^n \leq 0$. We now compute

$\dot{\alpha}^n_T$. We have $\tilde{v}^n_{8T} = \tilde{\beta}^n_{2T} = 0$, and thus

$$\dot{\alpha}^n_T = -r(1-\alpha^n_T)\alpha^n_T - \frac{\alpha^n_T \gamma^n_T \chi^n_T}{\sigma^2_X}\chi^n_T \beta^n_{1T} = \frac{\sigma^2_Y \psi \gamma^n_T \chi^n_T(-r_n\sigma^2_X + \gamma^n_T\chi^n_T)}{\sigma^2_X(\sigma^2_Y + \psi\gamma^n_T\chi^n_T)^2}, \tag{S.17}$$

so that $\dot{\alpha}^n_T \leq 0$ implies $\gamma^n_T\chi^n_T \leq r_n\sigma^2_X$. Since the minimum occurs at $t = T$, it follows that $\alpha^n_t \geq \alpha^n_T = \frac{\sigma^2_Y}{\sigma^2_Y + \psi\gamma^n_T\chi^n_T} \geq \frac{\sigma^2_Y}{\sigma^2_Y + \psi r_n \sigma^2_X}$, $t \in [0,T]$. Hence, $\alpha^n \uparrow 1$ uniformly as $n \to \infty$, and thus (γ^n, χ^n) converges uniformly to the solution (γ^*, χ^*) to their ODEs associated with $\alpha \equiv 1$, and by Lemma 3, $\gamma^*_T\chi^*_T > 0$. Thus $\gamma^n_T\chi^n_T \to \gamma^*_T\chi^*_T > 0$. But since $r_n\sigma^2_X \to 0$, we have $\gamma^n_T\chi^n_T > r_n\sigma^2_X$ for sufficiently large n, a contradiction.

S.2.5 Proof of Proposition 6

We analyze the public and no-feedback cases, and then we compare learning and payoffs.

Public Case

We look for an equilibrium of the form $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{3t}\theta$, where $M_t = \hat{M}_t$ is publicly known, with value function $V(\theta, m, t) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\theta^2 + v_{4t}m^2 + v_{5t}\theta m$.

The core (backward) system of ODEs is

$$(\dot{\beta}_{0t}, \dot{\beta}_{1t}, \dot{\beta}_{3t}, \dot{\gamma}_t) = (0, -\beta_{1t}\beta^2_{3t}\gamma^2_t/\sigma^2_Y, \beta_{1t}\beta^2_{3t}\gamma^2_t/\sigma^2_Y, \beta^2_{3t}\gamma^2_t/\sigma^2_Y).$$

with initial conditions $\beta_{00} = 0, \beta_{10} = -\frac{\psi\gamma_0}{\sigma^2_Y} \leq 0, \beta_{30} = 1$ and $\gamma_0 = \gamma^F \in (0, \gamma^o)$.

Define $\tilde{\psi} := \psi\gamma^o/\sigma^2_Y$ and $\tilde{T} := T\gamma^o/\sigma^2_Y$. It is easy to verify that the system has a solution for each root $\rho^{Pub} \in (0,1)$ of the cubic $g^{Pub}(\rho) := -\tilde{T}\tilde{\psi}\rho^2(1-\rho) + \rho(1+\tilde{T}) - 1 = 0$, where $(\beta_{0t}, \beta_{1t}, \beta_{3t}) = (0, \beta_{10}\gamma^F/\gamma_t, 1+\beta_{10}(1-\gamma^F/\gamma_t))$ where $\gamma_t = \frac{\gamma^F[\sigma^4_Y + t\psi(\gamma^F)^2]}{\sigma^4_Y - t\gamma^F(-\gamma^F\psi + \sigma^2_Y)}$ and $\gamma^F = \rho^{Pub}\gamma^o$. Such a root always exists since $g^{Pub}(0) < 0 < g^{Pub}(1)$, and hence an LME exists. Now the quadratic $g^{Pub\prime}(\rho)$ has roots only if $\tilde{\psi} \geq 3(1+\tilde{T})/\tilde{T} > 3$. Hence, when $\psi < \sigma^2_Y/\gamma^o$ (i.e., $\tilde{\psi} < 1$), g^{Pub} is strictly increasing, and the root, and hence the LME, is unique.

No Feedback Case

We look for an equilibrium with $a_t = \beta_0\mu + \beta_1 M_t + \beta_3\theta$, where $M_t = \mathbb{E}^1_t[\hat{M}_t]$, with value function $V(t, \theta, m) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\theta^2 + v_{4t}m^2 + v_{5t}\theta ml$. The backward system is

$$(\dot{\beta}_{0t}, \dot{\beta}_{1t}, \dot{\beta}_{3t}, \dot{\gamma}_t) = \big(0, -\alpha_t\beta_{1t}[\beta_{3t}\gamma_t - \beta_{1t}\gamma_t(1-\chi_t)]/\sigma^2_Y, \alpha_t\beta_{1t}\beta_{3t}\gamma_t/\sigma^2_Y, \alpha^2_t\gamma^2_t/\sigma^2_Y\big)$$

with initial conditions $\beta_{00} = 0$, $\beta_{10} = -\frac{\psi\gamma_0}{\sigma_Y^2+\psi\gamma_0\chi(\gamma_0)}$, $\beta_{30} = 1$, and $\gamma_0 = \gamma^F$, where $\chi(\gamma) := 1 - \gamma/\gamma^o$. Note that $\dot{\alpha}_t = 0$, so $\alpha = \alpha_0 = \frac{\sigma_Y^2}{\sigma_Y^2+\psi\gamma_0\chi(\gamma_0)}$, and it suffices to solve the γ-ODE.

By standard bounding arguments, the system has a solution. With algebra, one can show that there is one solution for each root $\rho^{NF} \in (0,1)$ of the quintic $g^{NF}(\rho) := \tilde{T}\rho - (1-\rho)[1 + \tilde{\psi}\rho(1-\rho)]^2$, with $\gamma_t = \frac{\gamma^F\sigma_Y^2}{\sigma_Y^2 - \gamma^F\bar{\alpha}^2 t}$, where $\gamma^F = \rho^{NF}\gamma^o$, and with $\alpha = \bar{\alpha} = \alpha_0$ being a constant as above. Such a root always exists since $g^{NF}(0) < 0 < g^{NF}(1)$, and hence an LME exists. When $\psi < \sigma_Y^2/\gamma^o$, it can be verified that g^{NF} is strictly increasing when it crosses zero in $(0,1)$; hence its root, and the LME, is unique.

Learning and Payoff Comparisons

The following lemma proves the claim about learning in the proposition.

Lemma S.7. *If $\tilde{\psi} \in (0,1]$, then there is more learning in the public case for all $T > 0$.*

Proof. Let $\rho^x = \gamma_T^x/\gamma^o \in (0,1)$, where γ_T^x is the terminal value of γ in the BVP of case $x \in \{\text{public, no feedback}\}$. When $\tilde{\psi} \in (0,1]$, these values are the unique roots of

$$
\begin{aligned}
0 = g^{NF}(\rho) &:= \rho\tilde{T} - (1-\rho)[1+\tilde{\psi}\rho(1-\rho)]^2 = \rho(1+\tilde{T}) - 1 - \tilde{\psi}\rho(1-\rho)^2[2+\tilde{\psi}\rho(1-\rho)] \\
0 = g^{Pub}(\rho) &:= \rho(1+\tilde{T}) - 1 - \tilde{\psi}\tilde{T}\rho^2(1-\rho),
\end{aligned}
$$

respectively. In particular, observe that $\rho^x > 1/(1+\tilde{T})$, $x \in \{\text{public, no feedback}\}$. Our goal is to show $\rho^{Pub} < \rho^{NF}$.

Now, using that $\rho^{Pub}(1+\tilde{T}) - 1 = \tilde{\psi}\tilde{T}(\rho^{Pub})^2(1-\rho^{Pub})$, we get that

$$
g^{NF}(\rho^{Pub}) = \frac{\tilde{\psi}(1-\rho^{Pub})}{\tilde{T}}\left\{\tilde{T}^2(\rho^{Pub})^2 - (1-\rho^{Pub})[2\rho^{Pub}\tilde{T} + \rho^{Pub}(1+\tilde{T}) - 1]\right\}.
$$

where $\frac{\tilde{\psi}(1-\rho^{Pub})}{\tilde{T}} > 0$. Thus, letting

$$
Q(\rho) := \tilde{T}^2\rho^2 - (1-\rho)[2\rho\tilde{T} + \rho(1+\tilde{T}) - 1] = \rho^2(\tilde{T}^2 + 3\tilde{T} + 1) - \rho(3\tilde{T}+2) + 1,
$$

it suffices to show that $Q(\rho^{Pub}) < 0$, as $g^{NF}(\rho) < 0$ if and only if $\rho < \rho^{NF}$.

Observe that the roots of Q are given by $\rho_- := \frac{(3-\sqrt{5})\tilde{T}+2}{2(\tilde{T}^2+3\tilde{T}+1)}$ and $\rho_+ := \frac{(3+\sqrt{5})\tilde{T}+2}{2(\tilde{T}^2+3\tilde{T}+1)}$, and that $\rho_- < \frac{1}{1+\tilde{T}} < \rho_+$. Consequently, it suffices to show that $g^{Pub}(\rho_+) > 0$: this ensures that $\rho^{Pub} < \rho_+$, and since $\rho^{Pub} > \frac{1}{1+\tilde{T}} > \rho_-$, this implies that $Q(\rho^{Pub}) < 0$.

Straightforward algebraic manipulation yields that $g^{Pub}(\rho_+) > 0$ if and only if

$$
\tilde{g}(\tilde{T},\tilde{\psi}) := 4(1+\tilde{T})[(3+\sqrt{5})\tilde{T}+2][\tilde{T}^2+3\tilde{T}+1]^2 - 8[\tilde{T}^2+3\tilde{T}+1]^3
$$

$$-\tilde{\psi}\tilde{T}^2[(3+\sqrt{5})\tilde{T}+2]^2[2\tilde{T}+(3-\sqrt{5})] > 0.$$

The constraint is tightest when $\tilde{\psi} = 1$, and $\tilde{g}(\tilde{T}, 1)$ can be written as $\tilde{T}\sum_{i=0}^{5} a_i\tilde{T}^i$ where all the $a_i > 0$. Hence, $\tilde{g}(\tilde{T}, \tilde{\psi}) > 0$ whenever $\tilde{T} > 0$ and $\tilde{\psi} \in (0, 1]$, concluding the proof. □

We now leverage Lemma S.7 to compare ex ante payoffs. To simplify expressions, we again rescale payoffs to remove the outside scalar factor of $\frac{1}{2}$. Let V^x denote the ex ante payoff to the politician in the case $x \in \{Pub, NF\}$. First,

$$\begin{aligned}
V^{Pub} &= \mathbb{E}_0\left[-\int_0^T (a_t-\theta)^2 dt - \psi M_T^2\right] \\
&= -\int_0^T \mathbb{E}_0\left[(\beta_{1t}M_t + [\beta_{3t}-1]\theta)^2\right] dt - \psi(\mu^2 + \gamma^o - \gamma_T) \\
&= -\int_0^T \left[(\beta_{3t}-1)^2\gamma^o + \beta_{1t}^2(\gamma^o-\gamma_t) + 2\beta_{1t}(\beta_{3t}-1)(\gamma^o-\gamma_t)\right] dt - \tilde{\psi}\sigma_Y^2(1-\rho^{Pub}).
\end{aligned}$$

Using the solutions for the coefficients and γ_t in terms of γ^F and carrying out the simplifications, we obtain $V^{Pub} = V^{Pub}(\rho^{Pub})$, where

$$V^{Pub}(\rho) := \sigma_Y^2\left\{-\tilde{\psi}(1-\rho) + \tilde{T}\tilde{\psi}\rho^2[-\tilde{\psi}(1-\rho)+1] + \ln\left(\frac{1-\rho}{\tilde{T}\rho}\right)\right\}.$$

In the no feedback case, note that $\mathbb{E}_0[M_t^2] = \mathbb{E}_0[(\chi_t\theta + (1-\chi_t)\mu)^2] = \mathbb{E}_0[\chi_t^2\theta^2] = \chi_t^2\gamma^o$. Hence, $\mathbb{E}_0[\hat{M}_t^2] = \mathbb{E}_0[(\hat{M}_t - M_t)^2] + \mathbb{E}_0[M_t^2] = \gamma_{2t} + \chi_t^2\gamma^o = \chi_t\gamma_t + \chi_t^2\gamma^o$.

Using $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{3t}\theta = \bar{a}\theta$, we now calculate

$$V^{NF} = \mathbb{E}_0\left[-\int_0^T (a_t-\theta)^2 dt - \psi(\chi_T\gamma_T + \chi_T^2\gamma^o)\right] = -(1-\bar{\alpha})^2\gamma^o T - \psi\chi_T(\gamma_T + \chi_T\gamma^o).$$

Expressing $\chi_T = 1 - \gamma_T/\gamma^o$, γ_T and $\bar{\alpha}$ in terms of $\gamma^F = \gamma_T$, we have $V^{NF} = V^{NF}(\rho^{NF})$, where $V^{NF}(\rho) := \sigma_Y^2\left\{-\tilde{\psi}^2\rho(1-\rho)^3 - \tilde{\psi}(1-\rho)\right\}$.

Lemma S.8. *For $\tilde{\psi} \in (0, 1]$, the sender is better off in the no feedback case than in the public case for all $T > 0$.*

Proof. We show that for such $\tilde{\psi}$, (i) $V^{Pub}(\rho^{Pub}) < V^{NF}(\rho^{Pub})$ and (ii) $V^{NF}(\rho)$ is increasing for $\rho \geq \rho^{Pub}$, so that $V^{Pub}(\rho^{Pub}) < V^{NF}(\rho^{NF})$.

Toward establishing (i), define $\tilde{V}(\rho) := V^{Pub}(\rho) - V^{NF}(\rho)$; we have

$$\tilde{V}(\rho) = \sigma_Y^2\left\{\tilde{T}\tilde{\psi}\rho^2[-\tilde{\psi}(1-\rho)+1] + \ln\left(\frac{1-\rho}{\tilde{T}\rho}\right) + \tilde{\psi}^2\rho(1-\rho)^3\right\},$$

and our first goal is to show $\tilde{V}(\rho^{Pub}) < 0$. Since $\ln(x) < x - 1$ for $x > 0$, we have

$$\tilde{V}(\rho) < \sigma_Y^2 \left\{ \tilde{T}\tilde{\psi}\rho^2[-\tilde{\psi}(1-\rho)+1] + \left[\frac{1-\rho}{\tilde{T}\rho} - 1\right] + \tilde{\psi}^2\rho(1-\rho)^3 \right\} = \frac{\sigma_Y^2}{\tilde{T}\rho}\tilde{V}_2(\rho),$$

where $\tilde{V}_2(\rho) := \tilde{T}^2\tilde{\psi}\rho^3[1-\tilde{\psi}(1-\rho)] + 1 - \rho(1+\tilde{T}) + \tilde{T}\tilde{\psi}^2\rho^2(1-\rho)^3$, and so it suffices to show $\tilde{V}_2(\rho^{Pub}) < 0$. Now the equation $g^{Pub}(\rho^{Pub}) = 0$ is equivalent to $\tilde{\psi} = -\frac{1-(1+\tilde{T})\rho}{\tilde{T}\rho^2(1-\rho)}|_{\rho=\rho^{Pub}}$; using this to eliminate $\tilde{\psi}$ and simplifying, we obtain $\tilde{V}_2(\rho^{Pub}) = -\frac{[\rho(1+\tilde{T})-1]^3}{\tilde{T}\rho^2}|_{\rho=\rho^{Pub}}$, which is strictly negative as $\rho^{Pub} > \frac{1}{1+\tilde{T}}$, establishing claim (i).

Toward claim (ii), differentiate

$$\frac{d}{d\rho}V^{NF}(\rho) = \sigma_Y^2 \left\{ -\tilde{\psi}^2[-3\rho(1-\rho)^2 + (1-\rho)^3] + \tilde{\psi} \right\} = \sigma_Y^2\tilde{\psi}\left\{ -\tilde{\psi}(1-\rho)^2(1-4\rho) + 1 \right\}.$$

The expression in braces is positive iff $h(\rho) := (1-\rho)^2(1-4\rho) < \frac{1}{\tilde{\psi}}$. Now for $\rho \in [0,1]$, $h(\rho)$ attains its maximum value of 1 at $\rho = 0$. Hence, if $\tilde{\psi} \leq 1$, the expression is positive for all $\rho \in (0,1)$ and we conclude that $V^{NF}(\rho)$ is increasing for all $\rho \geq \rho^{Pub}$.

Combining parts (i) and (ii) yields $V^{Pub}(\rho^{Pub}) < V^{NF}(\rho^{Pub}) < V^{NF}(\rho^{NF})$ as desired. $\square$

S.3 Section 5: Omitted Proofs

In this section we prove Lemma C.3 and Corollary C.2 from the main text.

Proof of Lemma C.3. Let $(x_t^i)_{t\in[0,T]}$ denote the solution to $IVP(y^i)$ from the statement of Lemma C.3. For each component $j \in \{1, \dots, n\}$, the triangle inequality implies

$$|x_{jt}^1 - x_{jt}^2| \leq |h_j(y_0^1) - h_j(y_0^2)| + \int_0^t |F_j(x_s^1, y_s^1) - F_j(x_s^2, y_s^2)| ds.$$

The mean value theorem and the facts that F_j is of class C^1 and X and Y are compact then yield that there exists $c^j \in \mathbb{R}_+$ such that

$$|x_{jt}^1 - x_{jt}^2| \leq |h_j(y_0^1) - h_j(y_0^2)| + c_j \int_0^t ||y_s^1 - y_s^2||_\infty ds + c_j \int_0^t ||x_s^1 - x_s^2||_\infty ds.$$

Letting $c = \max\{c_j : j = 1, ..., n\}$, we obtain

$$||x_t^1 - x_t^2||_\infty \leq ||\omega(y_0^1) - \omega(y_0^2)||_\infty + c\int_0^t ||y_s^1 - y_s^2||_\infty ds + c\int_0^t ||x_s^1 - x_s^2||_\infty ds.$$

Since $c > 0$ and $t \mapsto ||\omega(y_0^1) - \omega(y_0^2)||_\infty + c\int_0^t ||y_s^1 - y_s^2||_\infty ds$ is non-decreasing, Gronwall's inequality (Teschl, 2012, Lemma 2.7) implies that

$$\begin{aligned}
||x_t^1 - x_t^2|| &\leq e^{ct}\left(||\omega(y_0^1) - \omega(y_0^2)||_\infty + c\int_0^t ||y_s^1 - y_s^2||_\infty ds\right) \\
&\leq e^{cT}\left((||\omega(y_0^1) - \omega(y_0^2)||_\infty + cT \sup_{t\in[0,T]} ||y_t^1 - y_t^2||_\infty\right) \\
&= k_1||\omega(y_0^1) - \omega(y_0^2)||_\infty + k_2 \sup_{s\in[0,T]} ||y_s^1 - y_s^2||_\infty.
\end{aligned}$$

□

We now turn to Corollary C.2. Suppose the sender receives a terminal payoff of the form $\frac{1}{2}\psi(\hat{a}_T) = \psi_1\hat{a}_T + \psi_2\hat{a}_T^2$, where ψ_1, ψ_2 are constants. (An intercept is strategically irrelevant.) The following result gives sufficient conditions for the terminal game parameterized by (γ_T, χ_T) to have a unique equilibrium, continuously differentiable in those parameters. The bound $\underline{\psi}$ on the curvature, characterized in the proof of the lemma, ensures uniqueness by limiting strategic complementarities. Note that there are no restrictions on ψ_1.

Lemma S.9. *Under Assumptions 1 and 2, there exists* $\underline{\psi} \in (-\infty, 0) \cup \{-\infty\}$ *such that for all* $(\psi_1, \psi_2) \in \mathbb{R} \times (\underline{\psi}, 0]$ *there exists* $\vec{\beta}_T(\gamma_T, \chi_T)$ *continuously differentiable over* $[0, \gamma^o] \times [0, 1]$ *that, together with the receiver's myopic best reply, characterizes the unique Bayes Nash equilibrium of the terminal game parameterized by* (γ_T, χ_T). *Moreover,* $\alpha_T(\gamma_T, \chi_T) := \beta_{1T}(\gamma_T, \chi_T)\chi_T + \beta_{3T}(\gamma_T, \chi_T)$ *has the same sign as* $u_{a\theta}$, *and therefore the same sign as* α^m. *The threshold* $\underline{\psi}$ *depends on the parameter values as follows:*

- *If* $\hat{u}_{a\hat{a}}\hat{u}_{a\theta} = 0$, $\underline{\psi} = -\infty$.
- *If* $\hat{u}_{a\hat{a}}\hat{u}_{a\theta} \neq 0$, *then* $\underline{\psi}(\gamma^o) = -k/\gamma^o$, *where* $k > 0$ *is a constant.*

In particular, $|\underline{\psi}(\gamma^o)| \in \Omega(1/\gamma^o)$.

Proof. We first derive the system of equations that characterize any Bayes Nash equilibrium of the static game at time T. Given that (i) the receiver plays $\hat{a}_t = \delta_{0t} + \delta_{1t}\hat{M}_t + \delta_{2t}L_t$, where $\delta_{0t} = \hat{u}_{\hat{a}c} + \hat{u}_{a\hat{a}}\beta_{0t}$, $\delta_{1t} = \hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}(\beta_{3t} + \beta_{1t}\chi_t)$ and $\delta_{2t} = \hat{u}_{a\hat{a}}[\beta_{2t} + \beta_{1t}(1 - \chi_t)]$, and (ii) $M_t = \mathbb{E}_t[\hat{M}_t]$, all $t \in [0, T]$, imposing that the sender's strategy $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{2t}L_t + \beta_{3t}\theta$ satisfies the first-order condition on the right hand side of the HJB equation for times $t \in [0, T)$, we obtain the following equations:

$$\gamma_t\alpha_t v_{2t} = -\sigma_Y^2[u_{ac} + u_{a\hat{a}}\hat{u}_{\hat{a}c} - (1 - u_{a\hat{a}}\hat{u}_{a\hat{a}})\beta_{0t}] \tag{S.18}$$

$$\gamma_t\alpha_t v_{5t} = -\frac{\sigma_Y^2}{2}[u_{a\hat{a}}\hat{u}_{\hat{a}\theta} + u_{a\hat{a}}\hat{u}_{a\hat{a}}\alpha_t - \beta_{1t}] \tag{S.19}$$

$$\gamma_t\alpha_t v_{7t} = -\sigma_Y^2[u_{a\theta} - \beta_{3t}] \tag{S.20}$$
$$\gamma_t\alpha_t v_{9t} = -\sigma_Y^2[u_{a\hat{a}}\hat{u}_{a\hat{a}}\beta_{1t}(1-\chi_t) - \beta_{2t}(1-u_{a\hat{a}}\hat{u}_{a\hat{a}})]. \tag{S.21}$$

By continuity (wrt time) of the strategy, learning, and value function coefficients, (S.18)-(S.21) also hold at time $t = T$.

The sender's time-T expectation of the terminal payoff is

$$\mathbb{E}_T[\Psi(\hat{a}_T)] = \psi_1[\delta_{0T} + \delta_{1T}M_T + \delta_{2T}L_T] + \psi_2[\delta_{0T} + \delta_{1T}M_T + \delta_{2T}L_T]^2 + \psi_2\delta_{1T}^2\gamma_T\chi_T,$$

from which we obtain

$$v_{2T} = [\psi_1 + 2\psi_2(\hat{u}_{\hat{a}c} + \hat{u}_{a\hat{a}}\beta_{0T})](\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T) \tag{S.22}$$
$$v_{5T} = \psi_2(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T)^2 \tag{S.23}$$
$$v_{7T} = 0 \tag{S.24}$$
$$v_{9T} = 2\psi_2\hat{u}_{a\hat{a}}[\beta_{2T} + \beta_{1T}(1-\chi_T)](\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T). \tag{S.25}$$

For later use in our boundary value problem, we also note the terminal conditions $v_{6T} = \psi_2\delta_{2T}^2 = \psi_2\hat{u}_{a\hat{a}}^2[\beta_{2T} + \beta_{1T}(1-\chi_T)]^2$ and $v_{8T} = 0$.

Evaluating (S.18)-(S.21) at time $t = T$ and equating these with $(\gamma_T\alpha_T)$ times (S.22)-(S.25), respectively, we obtain

$$-\sigma_Y^2[u_{ac} + u_{a\hat{a}}\hat{u}_{\hat{a}c} - (1-u_{a\hat{a}}\hat{u}_{a\hat{a}})\beta_{0T}] = \gamma_T\alpha_T[\psi_1 + 2\psi_2(\hat{u}_{\hat{a}c} + \hat{u}_{a\hat{a}}\beta_{0T})](\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T) \tag{S.26}$$
$$-\frac{\sigma_Y^2}{2}[u_{a\hat{a}}\hat{u}_{\hat{a}\theta} + u_{a\hat{a}}\hat{u}_{a\hat{a}}\alpha_T - \beta_{1T}] = \psi_2\gamma_T\alpha_T(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T)^2 \tag{S.27}$$
$$-\sigma_Y^2[u_{a\theta} - \beta_{3T}] = 0 \tag{S.28}$$
$$-\sigma_Y^2[u_{a\hat{a}}\hat{u}_{a\hat{a}}\beta_{1T}(1-\chi_T) - \beta_{2T}(1-u_{a\hat{a}}\hat{u}_{a\hat{a}})] = 2\psi_2\gamma_T\alpha_T\hat{u}_{a\hat{a}}[\beta_{2T} + \beta_{1T}(1-\chi_T)](\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T). \tag{S.29}$$

It is immediate from (S.28) that in any solution, $\beta_{3T} = u_{a\theta}$; trivially, this is continuously differentiable in (γ_T, χ_T).

Next, we turn to α_T. Multiplying (S.27) through by $2\chi_T$, substituting $\beta_{1T}\chi_T = \alpha_T - \beta_{3T} = \alpha_T - u_{a\theta}$, and rearranging, (S.27) becomes

$$\underbrace{\sigma_Y^2[u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T - \alpha_T(1-u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T)] + 2\psi_2\gamma_T\chi_T\alpha_T(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T)^2}_{=:f(\alpha_T,\gamma_T,\chi_T)} = 0. \tag{S.30}$$

We construct $\underline{\psi}$ such that if $\psi \in (\underline{\psi}, 0]$, there exists a unique real α_T continuous in (γ_T, χ_T) over $[0, \gamma^o] \times [0, 1]$ that solves (S.30) and has the same sign as $u_{a\theta}$; from this, we construct β_1 solving (S.27) and in turn, β_0 and β_2 solving (S.26) and (S.29), respectively, all continuously differentiable in (γ_T, χ_T).

If $\psi_2 = 0$, $f(\cdot, \gamma_T, \chi_T)$ is linear and has unique root $\alpha_T(\gamma_T, \chi_T) := \frac{u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T}{1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T}$, which is well-defined and has the same sign as $u_{a\theta}$ for all $(\gamma_T, \chi_T) \in [0, \gamma^o] \times [0, 1]$ by Assumption 2. And clearly, it is continuous in (γ_T, χ_T) over this domain.

Hence, for the remainder of the proof, assume $\psi_2 < 0$. We consider two cases: $\hat{u}_{a\hat{a}} = 0$ and $\hat{u}_{a\hat{a}} \neq 0$. If $\hat{u}_{a\hat{a}} = 0$, $f(\cdot, \gamma_T, \chi_T)$ is linear and has unique root $\alpha_T(\gamma_T, \chi_T) := \frac{\sigma_Y^2(u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T)}{\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T) - 2\psi_2\gamma_T\chi_T\hat{u}_{\hat{a}\theta}^2} = \frac{\sigma_Y^2(u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T)}{\sigma_Y^2 - 2\psi_2\gamma_T\chi_T\hat{u}_{\hat{a}\theta}^2}$ which is well-defined, has the same sign as $u_{a\theta}$, and is continuous in (γ_T, χ_T) over $[0, \gamma^o] \times [0, 1]$. Specifically, the numerator of the expression defining α_T has the same sign as $u_{a\theta}$ by Assumption 2, and $\psi_2 \leq 0$ ensures that the denominator is positive. Thus, for $\hat{u}_{a\hat{a}} = 0$, the lemma holds with $\underline{\psi} = -\infty$, provided that the remaining variables are uniquely determined and continuously differentiable; we perform this step for both cases $\hat{u}_{a\hat{a}} = 0$ and $\hat{u}_{a\hat{a}} \neq 0$ after solving for α_T for the latter.

Now consider $\psi_2 < 0$ and $\hat{u}_{a\hat{a}} \neq 0$. If $\chi_T = 0$, then for all $\gamma_T \geq 0$, $f(\cdot, \gamma_T, \chi_T)$ is linear with intercept $\sigma_Y^2 u_{a\theta}$ and unique root $\alpha_T(\gamma_T, 0) := u_{a\theta}$.

Next, suppose $\chi_T \in (0, 1]$, $\psi_2 < 0$, and $\hat{u}_{a\hat{a}} \neq 0$. We establish a condition such that for all $(\gamma_T, \chi_T) \in [0, \gamma^o] \times (0, 1]$, $f(\cdot, \gamma_T, \chi_T)$ is strictly decreasing, and thus it has exactly one real root. Clearly this holds for $\gamma_T = 0$. If $\gamma_T > 0$, then $f(\cdot, \gamma_T, \chi_T)$ is cubic, and it satisfies $\lim_{\alpha_T \to +\infty} f(\alpha_T, \gamma_T, \chi_T) = -\infty$ and $\lim_{\alpha_T \to -\infty} f(\alpha_T, \gamma_T, \chi_T) = +\infty$. We calculate

$$\frac{\partial}{\partial \alpha_T} f(\alpha_T, \gamma_T, \chi_T) = -\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T) + 2\psi_2\gamma_T\chi_T(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T)(\hat{u}_{\hat{a}\theta} + 3\hat{u}_{a\hat{a}}\alpha_T), \quad \text{(S.31)}$$

which is concave and quadratic in α_T. The first term on the right hand side of (S.31) is negative by Assumption 2. The maximum value of the right hand side of (S.31), attained at $\alpha_T = -\frac{2\hat{u}_{\hat{a}\theta}}{3\hat{u}_{a\hat{a}}}$, is $-\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T) - \frac{2}{3}\hat{u}_{\hat{a}\theta}^2\psi_2\gamma_T\chi_T$. Thus

$$\frac{\partial}{\partial \alpha} f(\alpha_T, \gamma_T, \chi_T) \leq -\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\} - \frac{2}{3}\hat{u}_{\hat{a}\theta}^2\psi_2\gamma^o. \quad \text{(S.32)}$$

Define $\underline{\psi} = -\infty$ if $\hat{u}_{\hat{a}\theta} = 0$ and $\underline{\psi} := -\frac{\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\frac{2}{3}\hat{u}_{\hat{a}\theta}^2\gamma^o} < 0$ otherwise, where the value of k in the lemma is obvious. By construction, for all $\psi \in (\underline{\psi}, 0)$ and all $(\gamma_T, \chi_T) \in [0, \gamma^o] \times (0, 1]$, $f(\cdot, \gamma_T, \chi_T)$ is strictly decreasing and has a unique real root which we denote $\alpha_T(\gamma_T, \chi_T)$. Since $f(0, \gamma_T, \chi_T) = \sigma_Y^2[u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T]$ has the same sign as $u_{a\theta}$ by Assumption 2 and $f(\cdot, \gamma_T, \chi_T)$ is decreasing, $\alpha_T(\gamma_T, \chi_T)$ has the same sign as $u_{a\theta}$.

Having characterized $\alpha_T(\gamma_T, \chi_T) \neq 0$ on $[0, \gamma^o] \times [0, 1]$ for $(\psi_1, \psi_2) \in \mathbb{R} \times (\underline{\psi}, 0)$ for both cases $\hat{u}_{a\hat{a}} = 0$ and $\hat{u}_{a\hat{a}} \neq 0$, observe that $(\gamma_T, \chi_T) \mapsto \alpha_T(\gamma_T, \chi_T)$ is continuously differentiable by the implicit function theorem. We now characterize the remaining variables $(\beta_{0T}, \beta_{1T}, \beta_{2T})$. Given $\alpha_T(\gamma_T, \chi_T)$ as above, (S.27) uniquely determines $\beta_{1T}(\gamma_T, \chi_T)$ continuously differentiable. We now show there exist β_{0T} and β_{2T} solving (S.26) and (S.29), respectively, also continuously differentiable in (γ_T, χ_T). These equations are linear (and uncoupled) in β_{0T} and β_{2T}, respectively. Rearranging terms in (S.29) yields

$$\beta_{2T} \underbrace{\left[\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}) - 2\psi_2\gamma_T\alpha_T\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T)\right]}_{=:C(\gamma_T,\chi_T)}$$
$$= \hat{u}_{a\hat{a}}\beta_{1T}(1 - \chi_T)[2\psi_2\gamma_T\alpha_T(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T) + \sigma_Y^2 u_{a\hat{a}}].$$

Since collecting β_{0T} terms on the left side of (S.26) yields the same coefficient $C(\gamma_T, \chi_T)$, to establish existence it suffices to show that $C(\gamma_T, \chi_T) > 0$.

If $\alpha_T\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T) \geq 0$, we are done, since by Assumption 2, $\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}) > 0$, and by assumption, $\psi_2 \leq 0$ and $\gamma_T \geq 0$. Suppose now that $\alpha_T\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T) < 0$. Note that this implies $\hat{u}_{a\hat{a}} \neq 0$ and $\hat{u}_{\hat{a}\theta} \neq 0$, and by the definition of $\underline{\psi}$, $\psi_2 > \underline{\psi} = -\frac{\sigma_Y^2 \min\{1, 1-u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\frac{2}{3}\hat{u}_{\hat{a}\theta}^2\gamma^o}$. Thus, we have

$$\begin{aligned}
C(\gamma_T, \chi_T) &\geq \sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}) - 2\left[-\frac{\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\frac{2}{3}\hat{u}_{\hat{a}\theta}^2\gamma^o}\right]\gamma_T\alpha_T\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T) \\
&\geq \sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\} + \left[\frac{3\sigma_Y^2}{\hat{u}_{\hat{a}\theta}^2}\min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}\right]\alpha_T\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T) \\
&= \frac{\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\hat{u}_{\hat{a}\theta}^2}\left[\hat{u}_{\hat{a}\theta}^2 + 3\alpha_T\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T)\right] \\
&> \frac{\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\hat{u}_{\hat{a}\theta}^2}\left[\hat{u}_{\hat{a}\theta} + \frac{3}{2}\alpha_T\hat{u}_{a\hat{a}}\right]^2 \\
&\geq 0,
\end{aligned}$$

where the second line uses that $\gamma_T \leq \gamma^o$ and $\alpha_T\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_T) < 0$, and the fourth line uses that $\alpha_T\hat{u}_{a\hat{a}} \neq 0$ which implies $(\alpha_T\hat{u}_{a\hat{a}})^2 > 0$. Thus $C(\gamma_T, \chi_T) > 0$, so given α_T, (S.26) and (S.29) have unique solutions β_{0T} and β_{2T} which by inspection are continuously differentiable over the domain $[0, \gamma^o] \times [0, 1]$. This concludes the proof of the lemma statement.

For later use in our existence theorem, we note the following facts about the solution described above. First, β_{2T} carries a factor of $1 - \chi_T$ and (therefore) v_{6T} carries $(1 - \chi_T)^2$, while $v_{8T} = 0$. Hence, it is easy to perform a change of variables $(\tilde{\beta}_{2T}, \tilde{v}_{6T}, \tilde{v}_{8T}) = (\beta_{2T}/(1 - \chi_T), \gamma_T v_{6T}/(1 - \chi_T)^2, \gamma_T v_{8T}/(1 - \chi_T))$ as in the main text, all continuously differentiable in (γ_T, χ_T) and thus bounded over the compact domain $[0, \gamma^o] \times [0, 1]$. Second, after this change of variables, there exist nondecreasing functions $\eta, \bar{v} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\eta(\gamma^o), \bar{v}(\gamma^o) \to 0$ as $\gamma^o \to 0$ such that for all $(\gamma_T, \chi_T) \in [0, \gamma^o] \times [0, 1]$ and $i \in \{6, 8\}$, $||\vec{\beta}_T(\gamma_T, \chi_T) - \vec{\beta}^m(\chi_T)||_\infty \leq \eta(\gamma^o)$ and $|\tilde{v}_{iT}(\gamma_T, \chi_T)| \leq \bar{v}(\gamma^o)$. To see this, observe that the right hand sides of (S.26)-(S.29) converge uniformly to 0 as $\gamma_T \to 0$, and thus $\vec{\beta}_T(\gamma_T, \cdot)$ converges uniformly to $\vec{\beta}^m(\cdot)$ as $\gamma_T \to 0$. Similarly, it is easy to see that $\tilde{v}_{6T}(\gamma_T, \cdot) \to 0$ uniformly as $\gamma_T \to 0$, while $\tilde{v}_{8T}$ is identically zero. Setting $\eta(\gamma^o) := \sup\{||\vec{\beta}_T(\gamma_T, \chi_T) - \vec{\beta}^m(\chi_T)||_\infty : (\gamma_T, \chi_T) \in [0, \gamma^o] \times [0, 1]\}$ and $\bar{v}(\gamma^o) := \sup\{|\tilde{v}_{6T}(\gamma_T, \chi_T)| : (\gamma_T, \chi_T) \in [0, \gamma^o] \times [0, 1]\}$, we have that η and $\bar{v}$ are nondecreasing by construction, and they satisfy the inequalities and limit properties as claimed. □

Proof of Corollary C.2. We follow the same steps from before except for a few modifications, which we outline here. First, we note that the terminal values $(\beta_{1T}, \tilde{\beta}_{2T}, \beta_{3T}, \tilde{v}_{6T}, \tilde{v}_{8T})$ and $\alpha_T = \beta_{1T}\chi_T + \beta_{3T} \neq 0$ are now implicit C^1 functions of (γ_T, χ_T) over $[0, \gamma^o] \times [0, 1]$ given by Lemma S.9.[2]

In the '**Centering**' step, we replace the initial conditions for the backward ODEs of $(\vec{\beta}^c, \tilde{v}_6, \tilde{v}_8)$ with the difference $(\beta_{1T}, \tilde{\beta}_{2T}, \beta_{3T}, \tilde{v}_{6T}, \tilde{v}_{8T}) - (\beta_1^m, \tilde{\beta}_2^m, \beta_3^m, 0, 0)$ (suppressing dependence on (γ_T, χ_T)), and observe that the ODEs themselves do not change. Likewise, in the '**Auxiliary variable**' step, we modify the initial condition for the (backward) $\tilde{\alpha}$-ODE to be $\tilde{\alpha}_0 = \alpha_T(\gamma, \chi) \neq 0$ for all $(\gamma, \chi) \in [0, \gamma^o] \times [0, 1]$. Moreover, by the same comparison argument $\tilde{\alpha}$ does not change sign; but since α_T and α^m always have the same sign from the proof of Lemma S.9, it follows again that $\tilde{\alpha}/\alpha^m > 0$ from which we can find an interval of existence independent of $r \geq 0$. We also note that the argument showing that the solution to the boundary value problem satisfies $\alpha = \tilde{\alpha} \neq 0$ also remains unchanged.

Step 1 of the proof of Theorem C.1 is only modified in that the parameter used in our domain $\Lambda(\cdot)$ will be $\rho + K + \eta(\gamma^o)$ instead of $\rho + K$, to account for nonzero initial conditions for the centered variables. We elaborate on this parameter when discussing Step 3 below.

In Step 2 of the proof of Theorem C.1, we write for $i \in \{1, 2, 3\}$ and $j \in \{4, 5\}$

$$|\mathbf{b}_{it} - \mathbf{b}_{i0}| = \left| \int_0^t e^{-r \int_s^t \frac{\tilde{\alpha}_u}{\alpha_u^m} du} \hat{\gamma}_s h_i(\mathbf{b}_s - \mathbf{b}_0 + \mathbf{b}_0, \hat{\chi}_s) ds \right|$$
$$|\mathbf{b}_{jt} - \mathbf{b}_{j0}| = \left| \int_0^t e^{-\int_s^t (r + \hat{\gamma}_u R_j(\mathbf{b}_u, \hat{\chi}_u)) du} \hat{\gamma}_s h_j(\mathbf{b}_s - \mathbf{b}_0 + \mathbf{b}_0, \hat{\chi}_s) ds \right|.$$

Now from the end of the proof of Lemma S.9, there exist nondecreasing functions $\eta(\gamma^o)$ and $\bar{v}(\gamma^o)$ with $\eta(\gamma^o), \bar{v}(\gamma^o) \to 0$ as $\gamma^o \to 0$ such that for all $i \in \{1, 2, 3\}$, $|\mathbf{b}_{i0}| \leq \eta(\gamma^o)$ and for $j \in \{4, 5\}$, $|\mathbf{b}_{j0}| \leq \bar{v}(\gamma^o)$. Hence, when the bound $|\mathbf{b}_{is} - \mathbf{b}_{i0}| \leq K$ holds for

[2] Note that $\tilde{v}_{8T} = 0$ as before, but $\tilde{v}_{6T}$ can be nonzero.

all $i \in \{1,\dots,5\}$, we can bound $|h_i(\mathbf{b}_s - \mathbf{b}_0 + \mathbf{b}_0, \chi_s)| \leq h_i(K + \eta(\gamma^o), K + \bar{v}(\gamma^o))$ for scalars $h_i(K+\eta(\gamma^o), K+\bar{v}(\gamma^o))$ which are increasing in both arguments. Define $T(\gamma^o; K) := \min_{i\in\{1,\dots,5\}} \frac{K}{\gamma^o h_i(K+\eta(\gamma^o), K+\bar{v}(\gamma^o))}$. By repeating the arguments used in the proof of Lemma C.2, for all $T < T(\gamma^o; K)$, a solution to the modified version of (IVP$^{\text{bwd}}(\hat{\lambda})$) exists and satisfies $|\mathbf{b}_{it} - \mathbf{b}_{i0}| \leq K$ for all $t \in [0,T]$.

In Step 3, q is defined the same way as before, except now ω as in Lemma C.3 is the vector of new initial values for $(\vec{\beta}^c, \tilde{v}_6, \tilde{v}_8)$. Since ω is continuous, q remains continuous. Moreover,

$$\begin{aligned} q(\lambda) &= q(\lambda) - (\omega_1(\lambda_T), \omega_3(\lambda_T)) + (\omega_1(\lambda_T), \omega_3(\lambda_T)) \\ &= \underbrace{(\hat{\mathbf{b}}_1(\cdot;\lambda), \hat{\mathbf{b}}_3(\cdot;\lambda)) - (\omega_1(\lambda_T), \omega_3(\lambda_T))}_{||\cdot||_\infty \leq K} + \underbrace{(\mathrm{B}_1(\lambda_{(\cdot)}), \mathrm{B}_3(\lambda_{(\cdot)}))}_{||\cdot||_\infty \leq \rho} + \underbrace{(\omega_1(\lambda_T), \omega_3(\lambda_T))}_{||\cdot||_\infty \leq \eta(\gamma^o)}, \end{aligned}$$

and thus the triangle inequality yields $||q(\lambda)||_\infty \leq K + \rho + \eta(\gamma^o)$.

Step 4 goes through almost unchanged, except that (IVP$^{\text{fwd}}(q(\lambda))$) now takes as its input $q(\lambda)$, bounded by $K + \rho + \eta(\gamma^o)$ as above. Applying this bound to $|\dot{\boldsymbol{\lambda}}_{1t}|$ and $|\dot{\boldsymbol{\lambda}}_{2t}|$, it follows that the solution $\boldsymbol{\lambda}$ to (IVP$^{\text{fwd}}(q(\lambda))$) lies in $\Lambda(K+\rho+\eta(\gamma^o))$. By the same arguments as in the original Step 4, $q \mapsto \boldsymbol{\lambda}(q)$ is continuous, and the function g defined by $g(\lambda) = \boldsymbol{\lambda}(q(\lambda))$ is a continuous self-map on $\Lambda(K + \rho + \eta(\gamma^o))$. Schauder's Fixed Point Theorem then applies exactly as in Step 5. To conclude, we again define $T(\gamma^o)$ by maximizing $T(\gamma^o; K)$ over $K > 0$, and we note that $T(\gamma^o) \in \Omega(1/\gamma^o)$. □

S.4 Section 6: Omitted Proofs

The following lemma formalizes the assertion made under '**Private Value Environments**.'

Lemma S.10 (One-to-one mapping)**.** *Suppose that β_1 and β_3 are continuous and that $\delta_1 = \hat{u}_{\hat{a}a}\alpha_3$. If $\hat{u}_{\hat{a}a} \neq 0$, there are positive constants c_1, c_2 and d independent of γ^o such that*

$$\chi_t = \frac{c_1 c_2 (1 - [\gamma_t/\gamma^o]^d)}{c_1 + c_2[\gamma_t/\gamma^o]^d}.$$

Moreover, (i) $0 \leq \chi_t < c_2 < 1$ for all $t \in [0,T]$ and (ii) $c_2 \to 0$ as $\sigma_X \to 0$ and $c_2 \to 1$ as $\sigma_X \to \infty$. If instead $\hat{u}_{\hat{a}a} = 0$ or $\sigma_X = \infty$, $\chi_t = 1 - \gamma_t/\gamma^o$.

Proof. We first derive a candidate mapping. Suppose $\delta_1 = \hat{u}_{a\hat{a}}\alpha_3$. The χ-ODE is now

$$\dot{\chi}_t = \gamma_t \alpha_{3t}^2 \left(\frac{1-\chi_t}{\sigma_Y^2} - \frac{(\hat{u}_{a\hat{a}}\chi_t)^2}{\sigma_X^2} \right) =: -\gamma_t \alpha_{3t}^2 Q(\chi_t).$$

If $f : [0, \bar{\chi}) \to [0, \gamma^o]$, some $\bar{\chi} \in (0, 1]$, is differentiable and $f(\chi_t) = \gamma_t$ for all $t \geq 0$, then $f'(\chi_t)\dot{\chi}_t = \dot{\gamma}_t$. When $\alpha_{3t} \neq 0$, $\frac{f'(\chi_t)}{f(\chi_t)} = \frac{1/\sigma_Y^2}{Q(\chi_t)}$. Hence, we solve the ODE $\frac{f'(\chi)}{f(\chi)} = \frac{1/\sigma_Y^2}{Q(\chi)}$ for $\chi \in (0, \bar{\chi})$ where $f(0) = \gamma^o$.

To this end, let $c_2 := \frac{\sqrt{b^2+4(\hat{u}_{a\hat{a}})^2/[\sigma_X\sigma_Y]^2}-b}{2(\hat{u}_{a\hat{a}}/\sigma_X)^2}$ and $-c_1 := \frac{-\sqrt{b^2+4(\hat{u}_{a\hat{a}})^2/[\sigma_X\sigma_Y]^2}-b}{2(\hat{u}_{a\hat{a}}/\sigma_X)^2}$, where $b := 1/\sigma_Y^2$, be the roots of the convex quadratic Q above.

Clearly, $-c_1 < 0 < c_2$. Also, $c_2 \leq 1$ as $Q(1) \geq 0$; and when $\hat{u}_{a\hat{a}} \neq 0$, we have $c_2 < 1$. Thus, $\frac{1/\sigma_Y^2}{Q(\chi)} = -\frac{\sigma_X^2/\sigma_Y^2}{(\hat{u}_{a\hat{a}})^2(c_1+c_2)}\left[\frac{1}{\chi+c_1} - \frac{1}{\chi-c_2}\right]$ is well defined (and negative) over $[0, c_2)$ with $1/(\chi + c_1) > 0$ and $-1/(\chi - c_2) > 0$ over the same domain. We can then set $\bar{\chi} = c_2$ and solve $\int_0^{\chi} \frac{f'(s)}{f(s)} ds = -\frac{\sigma_X^2/\sigma_Y^2}{(\hat{u}_{a\hat{a}})^2(c_1+c_2)} \log\left(\frac{\chi+c_1}{c_2-\chi}\frac{c_2}{c_1}\right)$, which yields the decreasing function $f(\chi) = f(0)\left(\frac{c_1}{c_2}\right)^{1/d}\left(\frac{c_2-\chi}{\chi+c_1}\right)^{1/d}$, where $1/d = \sigma_X^2/[\sigma_Y^2(\hat{u}_{a\hat{a}})^2(c_1 + c_2)] > 0$. Imposing $f(0) = \gamma^o$ and inverting yields $\chi(\gamma) = f^{-1}(\gamma)$ as given in the lemma. Note that $\chi(\gamma^o) = 0$ and $\chi(0) = c_2$.

We now verify that $\chi(\gamma)$ satisfies the χ-ODE (even when $\alpha_3 = 0$). We have

$$\frac{d(\chi(\gamma_t))}{dt} = \frac{\alpha_{3t}^2\gamma_t}{\sigma_Y^2[c_1 + c_2(\gamma/\gamma^o)^d]^2} c_1 c_2 d[c_1 + c_2]\left(\frac{\gamma_t}{\gamma^o}\right)^d.$$

By construction, moreover, $c_1c_2 = c_1 - c_2 = \frac{\sigma_X^2}{\sigma_Y^2(\hat{u}_{a\hat{a}})^2}$, which follows from equating the first- and zero-order coefficients in $Q(\chi) = \hat{u}_{a\hat{a}}^2\chi^2/\sigma_X^2 + \chi/\sigma_Y^2 - 1/\sigma_Y^2 = \hat{u}_{a\hat{a}}^2(\chi - c_2)(\chi + c_1)/\sigma_X^2$. Thus, $dc_1c_2 = c_1 + c_2$. On the other hand,

$$\frac{[\hat{u}_{a\hat{a}}\chi(\gamma)]^2}{\sigma_X^2} = \frac{\hat{u}_{a\hat{a}}^2}{\sigma_X^2}\left[c_1c_2\frac{1-(\gamma/\gamma^o)^d}{c_1 + c_2(\gamma/\gamma^o]^d}\right]^2 = \frac{c_1^2(1-c_2)}{\sigma_Y^2}\left[\frac{1-(\gamma/\gamma^o)^d}{c_1 + c_2(\gamma/\gamma^o)^d}\right]^2$$

where we used that $c_1^2c_2^2/\sigma_X^2 = c_1^2(1-c_2)/\sigma_Y^2$ follows from $\hat{u}_{a\hat{a}}^2c_2^2/\sigma_X^2 = (1-c_2)/\sigma_Y^2$ by definition of c_2. Thus, the right-hand side of the χ-ODE evaluated at our candidate $\chi(\gamma)$ satisfies

$$\gamma_1\alpha_3^2\left(\frac{1-\chi}{\sigma_Y^2} - \frac{(\hat{u}_{a\hat{a}}\chi)^2}{\sigma_X^2}\right)\Bigg|_{\chi=\chi(\gamma)} = \frac{\alpha_3^2\gamma_1}{\sigma_Y^2}\left(1 - \chi - c_1^2(1-c_2)\left[\frac{1-(\gamma/\gamma^o)^d}{c_1 + c_2(\gamma/\gamma^o)^d}\right]^2\right).$$

Thus, using that $c_1c_2d = c_1 + c_2$ in our expression for $d(\chi(\gamma_t))/dt$, it suffices to show that

$$[c_1 + c_2]^2\left(\frac{\gamma_t}{\gamma^o}\right)^d = (1-\chi)[c_1 + c_2(\gamma/\gamma^o)^d]^2 - c_1^2(1-c_2)[1-(\gamma/\gamma^o)^d]^2.$$

Using that $\chi[c_1 + c_2(\gamma/\gamma^o)^d] = 1 - (\gamma/\gamma^o)$, it is easy to conclude that this equality reduces to three equations $0 = c_1^2 - c_1^2c_2 - c_1^2 + c_1^2c_2$, $(c_1 + c_2)^2 = 2c_1c_2 - c_1c_2(c_2 - c_1) + 2c_1^2(1 - c_2)$ and $0 = c_2^2 + c_1c_2^2 - c_1^2(1 - c_2)$, capturing the conditions on the constant, $(\gamma/\gamma^o)^d$ and $(\gamma/\gamma^o)^{2d}$, respectively. The first condition is trivially satisfied. As for the third, by the definition of c_1

and c_2 we have that $c_2^2/(1-c_2) = \sigma_X^2/(\hat{u}_{a\hat{a}}\sigma_Y)^2 = c_1^2/(1+c_1)$. Thus, $c_1^1(1-c_2) = c_2^2(1+c_1)$, and the result follows. By canceling common terms, the second condition is also a rearrangement of this identity. Thus, $\chi(\gamma)$ as postulated satisfies the χ-ODE; by uniqueness, $\chi = \chi(\gamma)$.

Finally, when $\hat{u}_{a\hat{a}} = 0$, we have that $\delta_1 \equiv 0$, and the χ-ODE reduces to $\dot{\chi} = \alpha_{3t}^2\gamma_t(1 - \chi_t)/\sigma_Y^2$, $\chi_0 = 0$. It is then easy to verify that $\chi(\gamma) = 1 - \gamma_t/\gamma^o$ satisfies the ODE, and we conclude again by uniqueness. □

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