Internet Appendix for "Leader-Follower Dynamics in Shareholder Activism"

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Abstract

In this Internet Appendix, we characterize equilibria for the case of a passive leader (see Remark 1 in Section 3.2), characterize non-PBS equilibria of the baseline model (Section 6), and prove Proposition 6. We also extend the model with a round of pre-game trading and show numerically that it can produce perfectly positively correlated positions as well as imperfectly negatively correlated positions entering our baseline leader-follower game, as described in Section 7.

I Passive leader

Proposition IA.1. *Suppose the leader is passive and $\rho \neq 0$. If $\rho > 0$, then a PBS equilibrium exists, and moreover, in any PBS equilibrium, $0 < \alpha_L < \alpha^K$, and the leader sells on average. If $\rho < 0$, there exists an equilibrium in which $\alpha_L < -\alpha^K$, $0 < \delta_L < \alpha^K$, and the leader still sells on average; and there is no equilibrium in which $\alpha_L > 0$. In both cases, the follower plays a gap strategy.*

Proof. Assume $\rho \neq 0$. The objectives of the activists are now

$$\text{Leader:} \qquad \sup_{\theta^L} \mathbb{E}[X_T^F X_T^L - P_1 \theta^L | X_0^L, \theta^L]$$

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$$\text{Follower:} \qquad \sup_{\theta^F} \mathbb{E}[X_T^F X_T^F - P_2\theta^F - \frac{1}{2}(X_T^F)^2 | X_0^F, \mathcal{F}_1, \theta^F].$$

For any conjectured linear strategies, price impacts are now

$$\Lambda_1 = \frac{\alpha_L \rho(1+\alpha_F)}{(\alpha_L^2\phi + \sigma^2)(1-\beta_F)} \tag{IA.1}$$

$$\Lambda_2 = \frac{\alpha_F(1+\alpha_F)\gamma_1^F}{\alpha_F^2\gamma_1^F + \sigma^2}, \tag{IA.2}$$

which differ from (A.6) and (A.11) only in that the component associated with the leader's terminal position is absent.

The follower's FOC is

$$\begin{aligned} 0 &= -\mathbb{E}_F[P_1 + \Lambda_2\{\Psi_2 - \mathbb{E}[\Psi_2|\mathcal{F}_1]\}] - \Lambda_2\theta^F + X_0^F + \theta^F & \text{(IA.3)} \\ &= -P_1 - \Lambda_2(\theta^F - [\alpha_F M_1^F + \beta_F P_1 + \delta_F \mu]) - \Lambda_2\theta^F + X_0^F + \theta^F, & \text{(IA.4)} \end{aligned}$$

and the leader's FOC is

$$\begin{aligned} 0 = &-\mathbb{E}_L[P_0 + \Lambda_1\{\Psi_1 - \mathbb{E}[\Psi_1]\}|\theta^L] - \theta\Lambda_1 + \mathbb{E}_L[X_T^F|\theta^L] \\ &+ (X_0^L + \theta^L)\frac{\partial \mathbb{E}_L[X_T^F|\theta^L]}{\partial \theta^L}. \end{aligned} \tag{IA.5}$$

Familiar arguments show that the strategy $\theta^F = \alpha_F(X_0^F - M_1^F)$, where $\alpha_F = \alpha_{F,1}(\alpha_L) = \sqrt{\frac{\sigma^2}{\gamma_1^F}}$, still satisfies the follower's FOC; that the follower's strategy has this characterization in any PBS equilibrium; and that the follower's strategy has a gap form in any linear equilibrium. Moreover, in this model, $M_1^F = P_1$, and $\beta_F = -\alpha_F$, and $\delta_F = 0$. It is easy to show that the leader's FOC implies the identity

$$\alpha_L = \frac{\sigma^2}{\phi\alpha_L} - \frac{\alpha_F}{1+\alpha_F}, \tag{IA.6}$$

where $\alpha_F = \alpha_{F,1}(\alpha_L)$, and the identity (A.17) for δ_L.

The SOCs reduce to

$$0 > 1 - 2\Lambda_2 = -\frac{1}{\alpha_F} \tag{IA.7}$$

$$0 > -2\Lambda_1(1-\beta_F) = -2\frac{\alpha_L\rho(1+\alpha_F)}{\alpha_L^2\phi + \sigma^2}, \tag{IA.8}$$

where again $\alpha_F = \alpha_{F,1}(\alpha_L)$.

The remainder of the proof analyzes separately the two cases $\rho > 0$ and $\rho < 0$.

$\rho > 0$ **case:** We first claim that there exist a $\alpha_L^+ \in (0, \alpha^K)$ solving (IA.6) and that it pins down a PBS equilibrium. As $\alpha_L \downarrow 0$, the RHS of (IA.6) tends to $+\infty$, and at $\alpha_L = \alpha^K$, the RHS is strictly less than α^K. Thus, by the intermediate value theorem, there exists a solution with $\alpha_L \in (0, \alpha^K)$. Moreover, there is no solution with $\alpha_L \geq \alpha^K$, since this would imply the RHS of (IA.6) is strictly less than $\frac{\sigma^2}{\phi\alpha_L} \leq \alpha_L$. Thus, for $\rho > 0$, $\alpha_L \in (0, \alpha^K)$ in any PBS equilibrium.

The follower's SOC (IA.7) is satisfied since $\alpha_F > 0$. The leader's SOC (IA.8) is also satisfied since $\alpha_L^+, \rho > 0$. Thus, the strategies characterized by α_L^+ and $\alpha_F = \alpha_{F,1}(\alpha_L^+)$ (along with $\beta_F = -\alpha_F$, $\delta_F = 0$, and δ_L as in (A.17) are part of a PBS equilibrium. The leader's expected trade $\mu(\alpha_L + \delta_L) = \mu(\alpha_L^+ - (\alpha^K)^2/\alpha_L^+)$ is negative since $\alpha_L^+ \in (0, \alpha^K)$, so the leader sells on average.

$\rho < 0$ **case:** We claim that there exists $\alpha_L^- \in (-\infty, -\alpha^K)$ solving (IA.6) and that it pins down a linear equilibrium. When $\alpha_L = -\alpha^K$, the RHS of (IA.6) equals $-\alpha^K - \frac{\alpha_F}{1+\alpha_F} < -\alpha_K$; and as $\alpha_L \to -\infty$, the RHS tends to a finite limit. Thus, by the intermediate value theorem, there exists $\alpha_L^- \in (-\infty, -\alpha^K)$ solving (IA.6). The follower's SOC (IA.7) is satisfied for the same reason as before, and the leader's SOC (IA.8) is satisfied since $\alpha_L^- < 0$. In such an equilibrium, the leader's expected trade is $\mu(\alpha_L + \delta_L) = \mu(\alpha_L^- - (\alpha^K)^2/\alpha_L^-) < 0$ since $\alpha_L^- \in (-\infty, -\alpha^K)$, so the leader still sells on average. Note that in this case it is impossible to have $\alpha_L > 0$, since it would not satisfy the (IA.8). □

II Results and Proofs for Section 6

This section analyzes non-PBS linear equilibria of the baseline model, as described in Section 6, and contains a proof of Proposition 6.

II.A Non-PBS Linear Equilibria

The results in the following proposition were referred to in Section 6.

Proposition IA.2. *(i) Positive correlation: If $\rho > 0$, then for sufficiently large $\sigma > 0$, there exists a linear equilibrium in which α_L and α_F are strictly negative.*

(ii) Perfect negative correlation: If $\rho = -\phi$, there is no linear equilibrium in which α_L and α_F have the same sign. A linear equilibrium in which $\alpha_L < 0 < \alpha_F$ exists for all $\sigma > 0$.

Proof. For part (i), we prove that for sufficiently large σ, there is a solution to (A.21) with $\alpha_L < 0$. We then check the conditions (A.18), (A.19), and $\phi(1 + \alpha_L) + \rho \neq 0$ and apply the "converse" part of Proposition A.1.

After a change of variables $x = \alpha_L/\sigma$ in (A.21), we obtain

$$-\sqrt{\frac{1+x^2\phi}{\phi + x^2(\phi^2-\rho^2)}} = \frac{\left(\frac{\rho+\phi}{\sigma} + \phi x\right)(x^2\phi - 1)}{\rho[1 - x\phi/\sigma - x^2\phi]}. \tag{IA.9}$$

When $x = -1/\sqrt{\phi}$, the right hand side vanishes, while the left hand side is strictly negative. Now choose σ sufficiently large that $\left(\frac{\rho+\phi}{\sigma} + \phi x\right) < 0$ for all $x \leq -1/\sqrt{\phi}$. Define $\alpha^\dagger$ to be the negative root of $\alpha_L(1+\alpha_L)\phi - \sigma^2$, and define $x^\dagger = \alpha^\dagger/\sigma < -1/\sqrt{\phi}$ to be the unique negative root of the denominator of (IA.9), where $x^\dagger \uparrow -1/\sqrt{\phi}$ as $\sigma \uparrow \infty$. The right hand side of (IA.9) is well-defined and continuous on $(x^\dagger, -1/\sqrt{\phi}]$ and moreover, it has limit $-\infty$ as $x \downarrow x^\dagger$. Thus, by the intermediate value theorem, there exists a solution x_L to (IA.9) in $(x^\dagger, -1/\sqrt{\phi}]$, and by the squeeze theorem, $\lim_{\sigma\uparrow\infty} x_L = -1/\sqrt{\phi}$. (By reversing the change of variables, one can recover α_L solving the leader's FOC.) Note that as $\sigma \uparrow \infty$, $x_F := \alpha_F/\sigma = -\sqrt{\frac{1+x^2\phi}{\phi+x^2(\phi^2-\rho^2)}} \to -\sqrt{\frac{2}{2\phi-\rho^2/\phi}} =: x_F^\infty$

To verify (A.18), note that this is equivalent to the condition $1 - x_L^2\phi - 2x_L\left(\frac{\rho+\phi}{\sigma} + \rho x_F\right) \leq 0$. As $\sigma \uparrow +\infty$, the left hand side has limit $1 - 1 - 2(-1/\sqrt{\phi})\rho x_F^\infty = 2\rho x_F^\infty/\sqrt{\phi} < 0$, so (A.18) is satisfied for sufficiently large σ.

As for (A.19), using that $\alpha_{F,2} < 0$, it suffices to show that

$$\sigma^2[x_L^2(\phi^2-\rho^2) + x_L\sigma\rho + (\phi+\rho)] \leq 0.$$

Recall that x_L has finite limit as $\sigma \to +\infty$, so the dominating term is $\sigma^3 x_L\rho < 0$. We conclude that (A.19) is satisfied for sufficiently large σ.

Finally, observe that since the left side of (IA.9) is nonzero, at our solution the right side is also nonzero, and thus $\frac{\rho+\phi}{\sigma} + \phi x_L = \frac{1}{\sigma}[\phi(1+\alpha_L)+\rho] \neq 0$. Hence Proposition A.1 applies, giving us existence for large σ.

For part (ii), we begin with the observation that for $\rho = -\phi$, (A.19) becomes

$$\sigma^2\phi\alpha_F\alpha_L \leq 0. \tag{IA.10}$$

Hence, there is no equilibrium in which α_F and α_L are both strictly positive or both strictly negative, and (9)-(10) imply $\alpha_L \neq 0$ and $\alpha_F \neq 0$.

We now establish the existence of an equilibrium with $\alpha_L < 0 < \alpha_F$. Note that for $\rho = -\phi$, as long as $\alpha_L \neq 0$ (which must hold in any equilibrium), the condition $\phi(1+\alpha_L)+\rho \neq 0$ is satisfied. When $\rho = -\phi$ and $\alpha_F = \alpha_{F,1}$, (A.20) simplifies to

$$\sqrt{\sigma^2/\phi + \alpha_L^2} = \alpha_L\frac{\alpha_L^2\phi - \sigma^2}{\alpha_L(1+\alpha_L)\phi - \sigma^2}. \tag{IA.11}$$

In particular, an equilibrium with $\alpha_F = \alpha_{F,1}$ exists if and only if there exists α_L satisfying (IA.11) such that both SOCs are satisfied. Now the left hand side of (IA.11) is positive, while the right hand side vanishes at $\alpha_L = -\sigma/\sqrt{\phi}$, has limit $+\infty$ as $\alpha_L \downarrow \alpha^\dagger$, and is continuous on $(\alpha^\dagger, -\sigma/\sqrt{\phi})$, where $\alpha^\dagger$ was previously defined as the negative root of $\alpha_L(1+\alpha_L)\phi - \sigma^2$, and recall that $\hat{\alpha}$ is the positive root. Thus, (IA.11) has a solution in this interval. We finally check (A.18), which is now $\sigma^2 - \alpha_L^2\phi + 2\alpha_L\phi\alpha_F \leq 0$. This is satisfied since $\alpha_L < -\sigma/\sqrt{\phi}$ implies $\sigma^2 - \alpha_L^2\phi < 0$, and clearly $2\alpha_L\phi\alpha_F < 0$. Since α_F and α_L have opposite signs, (A.19) is satisfied. Hence, existence follows from Proposition A.1. □

II.B Proof of Proposition 6

Since Proposition A.2 establishes existence and uniqueness for all $\sigma > 0$ when $\rho = 0$, assume $\rho \neq 0$. We will show that for sufficiently small $\sigma > 0$, there is a unique pair (α_L, α_F) satisfying (A.13), (A.29), (A.18), and (A.19). Further, we will show that $\phi(1+\alpha_L) + \rho \neq 0$, so existence follows from Proposition A.1.

In any equilibrium, (α_L, α_F) must solve (A.29). By squaring both sides of this equation, using (A.13), and multiplying through by the nonzero denominator, we get (A.31). Now as $\sigma \to 0$, the coefficients of the polynomial Q converge to those of

$$Q^{\sigma=0}(\alpha_L) := -\alpha_L^6\phi^2[\rho + \phi + \alpha_L\phi]^2(\phi^2 - \rho^2), \tag{IA.12}$$

which has a root of multiplicity 6 at 0 and of multiplicity 2 at $-\frac{\rho+\phi}{\phi}$.

By Lemma A.2, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\sigma \in (0, \delta)$, Q has 6 complex roots within distance ϵ of 0 and 2 complex roots within ϵ of $-\frac{\rho+\phi}{\phi}$. For ϵ sufficiently small that these neighborhoods do not intersect, and δ chosen accordingly, let $\alpha_1, \ldots, \alpha_6$ denote the 6 roots near 0, and let α_7 and α_8 denote the roots near $-\frac{\rho+\phi}{\phi}$. We maintain these assumptions on ϵ and δ throughout the proof.

The following lemma rules out α_7 and α_8 from being part of an equilibrium.

Lemma IA.1. *For sufficiently small $\sigma > 0$, each of α_7 and α_8 is either complex or otherwise fails (A.18).*

Proof. The left side of (A.18) is continuous in (σ, α_L) at $\left(0, -\frac{\rho+\phi}{\phi}\right)$, where it evaluates to $(\phi+\rho)^2/\phi > 0$. Hence, choosing $\epsilon > 0$ sufficiently small, and $\delta > 0$ sufficiently small as described before the lemma, if either α_7 or α_8 is real, it fails (A.18). □

Remark 1. *Having ruled out α_7 and α_8, note that if σ is sufficiently small, then for any real $\alpha_L \in \{\alpha_1, \ldots, \alpha_6\}$, $\rho + \phi + \alpha_L\phi \neq 0$. This fact is useful two fold: (i) this criterion appears in* *the sufficiency part of Proposition A.1, and (ii) due to (A.29), using that $\rho \neq 0$ and $\alpha_{F,1} \neq 0$ and $\alpha_{F,2} \neq 0$ for α_L real, we have $\sigma^2 - \alpha_L(1+\alpha_L) \neq 0$ for sufficiently small σ for α_L real. Thus, any real solution to (A.31) solves (A.30).*

We can now rule out equilibria in which $\alpha_F = \alpha_{F,2}$, as these fail the follower's second order condition when σ is sufficiently small. To do so, we use asymptotic properties of the roots of (A.31) as $\sigma \to 0$.

It is useful to define a change of variables $z = \alpha_L/\sigma$ in (A.31) and divide through the resulting equation by σ^6, obtaining an equivalent equation

$$0 = \tilde{Q}(z,\sigma) := \sigma H(z) + F(z), \tag{IA.13}$$

where $H(z)$ is a polynomial of degree 8 and where $F(z)$ is a polynomial independent of σ that has the form $c_6 z^6 + c_4 z^4 + c_2 z^2 + c_0$.[1] For each $i \in \{1, 2, \ldots, 6\}$, define $z_i = \alpha_i/6$.

Lemma IA.2. *F has 6 distinct roots, denoted $\hat{z}_1, \ldots, \hat{z}_6$, of which exactly two are positive, two are negative, and two are complex. As $\sigma \to 0$, $z_1, \ldots, z_6$ converge to $\hat{z}_1, \ldots, \hat{z}_6$.*

Proof. We first characterize the roots of F. Consider the cubic polynomial $G(y) = c_6 y^3 + c_4 y^2 + c_2 y + c_0$, where $F(y) = G(y^2)$. We have $G(0) < 0$ and $\lim_{y\to-\infty} G(y) = +\infty$, so G has a negative root. Also, we have $\lim_{y\to+\infty} G(y) = -\infty$ and $G(1/\phi) = 2\rho^2\phi > 0$, so G has two distinct positive roots: one in $(0, 1/\phi)$ and one in $(1/\phi, +\infty)$. Since G is cubic, there are no other roots (real or complex). Now the negative root of G corresponds to two distinct complex roots of F, and the positive roots of G each correspond to both one positive and one negative root of F, all distinct.

We now turn to the convergence claim in the lemma. Next, set $K = 1 + \max_{i\in\{1,\ldots,6\}} |\hat{z}_i|$, and define a compact set $\mathcal{K} = \{z \in \mathbb{C} : |z| \leq K\}$. By definition, all roots of F lie in $\mathcal{K}$. Further, note that on $\mathcal{K}$, for any sequence $(\sigma_n)_{n\in\mathbb{N}}$ with $\sigma_n \downarrow 0$, the sequence $(\tilde{Q}(\cdot, \sigma_n))_{n\in\mathbb{N}}$ of functions defined on $\mathcal{K}$ is equicontinuous and converges pointwise to F since $\sigma H(z)$ vanishes; thus, by the Arzela-Ascoli theorem, the sequence converges uniformly to F on $\mathcal{K}$.

Choose $\overline{\eta} > 0$ less than 1 and less than the minimum distance between any $\hat{z}_i$ and $\hat{z}_j$, where $i, j \in \{1, \ldots, 6\}$ and $i \neq j$. Then for all $\eta \in (0, \overline{\eta})$, for each $i \in \{1, \ldots, 6\}$, 0 is the unique value of $t \in (1-\eta, 1+\eta)$ such that $0 = F(t\hat{z}_i)$. Further, $F(t\hat{z}_i)$ takes opposite signs at $t = 1+\eta$ and $t = 1-\eta$. By uniform convergence, for each such η, it holds that for all sufficiently small $\sigma > 0$, and for all $i \in \{1, \ldots, 6\}$, $\tilde{Q}((1+\eta)\hat{z}_i, \sigma)$ and $\tilde{Q}((1-\eta)\hat{z}_i, \sigma)$ have the same signs as $F((1+\eta)\hat{z}_i)$ and $F((1-\eta)\hat{z}_i)$, respectively; thus, for all sufficiently small $\sigma > 0$, there exists $t_i(\sigma)$ in $(1-\eta, 1+\eta)$ such that $\tilde{Q}(t_i(\sigma)\hat{z}_i, \sigma) = 0$, and therefore,

[1] In particular, $F(z) = -z^6(\phi-\rho)\phi^2(\phi+\rho)^3 + z^4\phi[-2\rho^4 - 4\rho^3\phi + 2\rho\phi^3 + \phi^4] + z^2(\rho^2 + \rho\phi + \phi^2)^2 - \phi(\rho+\phi)^2$.

$\{z_1, \ldots, z_6\} = \{t_1(\sigma), \ldots, t_6(\sigma)\}$. Relabelling so that $z_i = t_i(\sigma)$, we have $z_i \to \hat{z}_i$ for each $i \in \{1, \ldots, 6\}$. □

We now analyze the follower's SOC.

Lemma IA.3. *If $\sigma > 0$ is sufficiently small, then (i) there is no equilibrium in which $\alpha_F = \alpha_{F,2}$, and (ii) for $\alpha_F = \alpha_{F,1}$, (A.19) is satisfied for all real roots of Q among $a_1, \ldots, a_6$.*

Proof. Having ruled out equilibria in which $\alpha_L \in \{\alpha_7, \alpha_8\}$ (when $\sigma > 0$ is small), we show that for $\alpha_F = \alpha_{F,2}$ and for sufficiently small $\sigma > 0$, (A.19) fails for all real roots among $\alpha_1, \ldots, \alpha_6$. By Lemma IA.2, each α_i/σ, $i \in \{1, \ldots, 6\}$, converges to a finite nonzero limit $\hat{z}_i$. Hence, for sufficiently small $\sigma > 0$, if $\alpha_L = \alpha_i$, for some $i \in \{1, \ldots, 6\}$ is real, the factor in square brackets in (A.19) is bounded below by

$$\begin{aligned}\alpha_i^2(\phi^2 - \rho^2) + \sigma^2(\phi + \rho) - |\alpha_i \rho|\sigma^2 &\geq \alpha_i^2(\phi^2 - \rho^2) + \sigma^2(\phi + \rho) - |\rho z_i|\sigma^3 \\ &= \sigma^2(z_i^2(\phi^2 - \rho^2) + \phi + \rho - |\rho z_i|\sigma),\end{aligned}$$

where $z_i^2(\phi^2 - \rho^2) + \phi + \rho - |\rho z_i|\sigma \to \hat{z}_i(\rho^2 - \rho^2) + \phi + \rho > 0$. Since $-\alpha_{F,2} > 0$, this implies that (A.19) fails.

For $\alpha_F = \alpha_{F,1}$, the same bound above holds, but since $-\alpha_{F,1} < 0$, (A.19) is satisfied. □

From the proof of Proposition A.4, any equilibrium value of α_L must solve (A.20) (with $\alpha_F = \alpha_{F,1}$) or (A.21)(with $\alpha_F = \alpha_{F,2}$). By Lemma IA.3 part (i), α_L must solve (A.20).

We now turn to the leader's SOC.

Lemma IA.4. *If $\sigma > 0$ is sufficiently small, then (i) there is no equilibrium in which $\alpha_L \leq 0$, and (ii) if $\alpha_L > 0$ is a real root of (A.31) and $\alpha_F = \alpha_{F,1}$, then (A.18) is satisfied.*

Proof. For part (i), we only need to consider the roots $\alpha_1, \ldots, \alpha_6$, since for sufficiently small σ α_7 and α_8 cannot be part of an equilibrium by Lemma IA.1. By Lemma IA.3, we further only need to consider $\alpha_F = \alpha_{F,1}$, for which (A.18) becomes

$$\sigma^2 - \alpha_L^2\phi - 2\alpha_L\left(\rho + \phi + \rho\sigma\sqrt{\frac{\sigma^2 + (\alpha_L/\sigma)^2\sigma^2\phi}{\phi + (\alpha_L/\sigma)^2(-(\rho)^2 + (\phi)^2)}}\right) \leq 0. \tag{IA.14}$$

Clearly, this is violated if $\alpha_L = 0$. And since $\alpha_L \to 0$ in proportion to σ by Lemma IA.2, for small σ, the dominating term is $-2\alpha_L(\rho + \phi)$, which is positive (violating (IA.14)) if $\alpha_L < 0$.

For part (ii), we again only need to consider the roots $\alpha_1, \ldots, \alpha_6$, since for sufficiently small σ, α_7 and α_8 are not positive real numbers as they converge to $-\frac{\rho+\phi}{\phi}$. Following the same calculation above, for sufficiently small σ, the left hand side of (A.18) has the same sign as $-2\alpha_L(\rho+\phi)$, which is negative for $\alpha_L > 0$, satisfying (A.18). □

In light of Lemma IA.4, we use Lemma IA.2 to show that for sufficiently small $\sigma > 0$, there is exactly one positive solution to (A.20), and thus one equilibrium candidate. We establish this in the following lemma:

Lemma IA.5. *For sufficiently small $\sigma > 0$, equation (A.31) has exactly two positive roots, one solving (A.20) and the other solving (A.21).*

Proof. Any (positive) solution to (A.20) or (A.21) must be a (positive) root of (A.31). From the proof of Proposition A.4, (A.31) has *at least* two positive roots, one for each equation (A.20) and (A.21), so it suffices to show that these are the only two positive roots of (A.31). Using the change of variables $z = \alpha_L/\sigma$, $\tilde{Q}(\cdot,\sigma)$ has at least two positive real roots for all sufficiently small σ. But $\tilde{Q}(\cdot,\sigma)$ cannot have more than two positive roots for all sufficiently small σ. To see this, recall that for small σ, α_7 and α_8 are complex or negative, so any positive roots must be among $\alpha_1,\ldots,\alpha_6$. And if there were more than two such positive roots, then by Lemma IA.2, F would have more than two nonnegative roots, a contradiction. Mapping back to $\alpha_L = z\sigma$, this implies that (A.31) has exactly two roots for sufficiently small σ, (A.20) and (A.21) each have exactly one. □

From Lemmas IA.3, IA.4, and IA.5, for sufficiently small $\sigma > 0$, there is exactly one pair (α_L, α_F) solving (A.13), (A.16), (A.19), and (A.18), and thus at most one equilibrium. By Remark 1, we can invoke the "converse" part of Proposition A.1, establishing existence.

III Endogeneizing Initial Positions

In this section, we analyze an extension of the model with pre-game trading and show numerically that it can endogenize perfect positive correlation and imperfect negative correlation, as mentioned in Section 7.

Setup

There are two identical traders $i = 1, 2$ who start with no ownership on the firm's stock and simultaneously place orders θ_i. A market maker (MM) observes total order flow

$$\Psi_0 = \theta_1 + \theta_2 + \sigma Z_0,$$

where $\sigma > 0$ is a known constant (the same as in the leader-follower game) and $Z_0 \sim N(0, 1)$, and executes at a price P_{pre}. Suppose the firm's value has an exogenous additive component $v \sim N(0, \sigma_v^2)$, and the each agent observes a noisy signal

$$s_i = v + \epsilon_i$$

where ϵ_1, ϵ_2 are jointly normal with mean 0 and the following covariance matrix: $\begin{pmatrix} \sigma_\epsilon^2 & \rho_\epsilon \sigma_\epsilon^2 \\ \rho_\epsilon \sigma_\epsilon^2 & \sigma_\epsilon^2 \end{pmatrix}$.

After this round of trading, we assume that v is publicly revealed.[2] Then, with probability q, it is publicly revealed that there are activism opportunities at the target firm, meaning that our leader-follower game is played, and firm value (per share) is the sum of v and the players' efforts; in this case, we assume that the roles of leader and follower are assigned to the players with equal probabilities. Finally, with complementary probability $1 - q$ the game ends (the leader-follower sub-game does not arise).

We focus on *symmetric linear equilibria*, where

(i) Traders trade in the pregame according to symmetric strategies αs_i;

(ii) The MM uses a linear pricing rule $P_{pre} = \phi + \Lambda_0 \Psi_0$ in the pregame;[3]

(iii) Traders follow optimal (role-specific) strategies in the leader-follower game, and the MM uses a linear pricing rule.

Overview of results We show that this framework can produce both positive and negative correlation by adjusting the correlation in the noise of the players' pregame signals.

- Positive correlation: We specialize to the case $\rho_\epsilon = 1$ and numerically establish the existence of an equilibrium. From the perspective of the market maker, players' initial positions entering the leader-follower game have perfect positive correlation. The restriction to $\rho_\epsilon = 1$ is purely to simplify the expressions involved; by continuity the result extends to ρ_ϵ in a neighborhood of 1.

- Negative correlation: We specialize to the case $\rho_\epsilon = 0$, i.e. players receive conditionally i.i.d. signals of v, and establish the existence of an equilibrium. From the perspective

[2] We consider the perfect revelation of the exogenous component not because we cannot carry two forms of private information (block sizes and fundamental value), but because it simplifies the task of generating positive/negative correlation while fitting 100% in our baseline model. Even if we consider two-dimensional private information, activists using linear strategies for the both pieces of private information; hence this does not effect our point of generating the initial correlation structure.

[3] We use P_{pre} to distinguish from P_0, the expected firm value at the beginning of the leader-follower game (after v has been revealed).

of the MM, conditional on the pregame order flow and v, players' positions now have (imperfect) negative correlation.

The solution of this model is more complicated, mainly due to two additional forces. First, deviations in the pre-game lead to private information that can be payoff-relevant in the sequential game for our players—the continuation game changes after deviations. Specifically, with perfectly correlated signals, players must use both their signal and their actual position resulting from pre-game trading to best respond in the continuation game; they use the signal to forecast the other player's position (who they assume is on path), but after deviations in the pre-game, the player's own position is decoupled from the signal. With conditionally i.i.d. signals, however, the pre-game signals are only needed in the pre-game, since after the revelation of v, a player's own signal becomes irrelevant for forecasting the other's signal or position—the player's own position is the only relevant private information in the continuation game, on and off path.

Second, there is a non-trivial fixed point at play: the coefficient in the trading strategy in the pre-game shapes the degree of correlation in initial blocks in our sequential game which, via continuation payoffs, in turn matters for the determination of the aforementioned coefficient itself in pregame.

Positive Correlation

In this section, assume $\rho_\epsilon = 1$. We reduce the problem of existence of a (symmetric linear) equilibrium to a fixed point equation in α, which we solve numerically. This equilibrium generates perfectly correlated positions from the perspective of the market maker.

Belief updating in the pregame Under perfectly correlated signals, player i knows $s_{-i} = s_i$. Now conjectured equilibrium strategies and order flow Ψ_0 the market maker's updated beliefs are

$$\mu_v := \mathbb{E}[v|\Psi_0] = \frac{2\alpha\sigma_v^2}{4\alpha^2(\sigma_v^2 + \sigma_\epsilon^2) + \sigma_0^2}\Psi_0$$
$$\mu_\theta := \mathbb{E}\left[\theta^i|\Psi_0\right] = \frac{2\alpha(\sigma_v^2 + \sigma_\epsilon^2)}{4\alpha^2(\sigma_v^2 + \sigma_\epsilon^2) + \sigma_0^2}\Psi_0.$$

Recall that in the leader follower game, with prior mean μ, expected firm value is $(2 + \alpha_L + \delta_L)\mu$. Hence, given Ψ_0, the MM sets price

$$P_{pre} = \mu_v + q(2 + \alpha_L + \delta_L)\mu_\theta.$$

At the end of the pregame, players update based on (Ψ_0, v) due to v becoming public. Because of the presence of noise traders, deviations are hidden and hence each player correctly assumes the other is on path; thus player i believes $X_0^{-i} = \alpha s_i$ with probability 1.

The MM assumes both players are on path and have identical positions. The MM's posterior mean given (Ψ_0, v) is

$$\begin{aligned}\mu_X := \mathbb{E}[X_0^i | v, \Psi_0] &= \alpha v + \frac{\text{Cov}(\Psi_0, X_i|v)}{\text{Var}(\Psi_0|v)}(\Psi_0 - 2\alpha v) \\ &= \alpha v + \frac{\text{Cov}(2\alpha(v+\epsilon) + \sigma Z_0, \alpha(v+\epsilon)|v)}{\text{Var}(2\alpha(v+\epsilon) + \sigma Z_0|v)}(\Psi_0 - 2\alpha v) \\ &= \alpha v + \frac{2\alpha^2\sigma_\epsilon^2}{4\alpha^2\sigma_\epsilon^2 + \sigma^2}(\Psi_0 - 2\alpha v).\end{aligned}$$

The posterior variance is

$$\phi := \text{Var}(X_0^i | v, \Psi_0] = \frac{\alpha^2\sigma_\epsilon^2\sigma^2}{4\alpha^2\sigma_\epsilon^2 + \sigma^2} \tag{IA.15}$$

Due to perfectly correlated signals, the traders do not use (Ψ_0, v) to update beliefs about each other's signals and positions.

Best response problems in the leader-follower game In this subsection, we solve the players' best response problems in the leader-follower game after arbitrary histories of the pregame.

In a conjectured equilibrium, the relevant state variables entering the leader-follower game are (X_0^i, s_i, μ_X, v), where $\mu_X := \mathbb{E}[X_0^i|\Psi_0, v]$. A few comments are in order:

1. Although the prior expectation of s_i is just v, the public posterior expectation "μ_s" about s_i given (Ψ_0, v) is not v; higher Ψ_0 is indicative of higher errors ϵ_i.

2. However, on the path of play of the pre-game, $X_0^i = \alpha s_i$, and the MM assumes players are on path, so μ_X is a sufficient statistic for μ_s: $\mu_s = \mu_X/\alpha$.

3. Also on the path of play, X_0^i is a sufficient statistic for s_i, but since players can deviate in the pre-game, s_i is a relevant state entering the leader-follower game.

4. All first-order beliefs and higher-order beliefs about (X_0^i, s_i) can be written in terms of (X_0^i, s_i, μ_X, v).

Write the expanded strategies of the players in the leader-follower game as

$$\theta_L = \hat{\alpha}_L X_0^L + \delta_L \mu + \hat{\nu}_L s_L$$

$$\theta_F = \hat{\alpha}_F(X_0^F - M_1^F) + \hat{\nu}_F(s_F - M_1^F/\alpha) = \hat{\alpha}_F X_0^F + \hat{\nu}_F s_F + \beta_F(P_1 - v) + \delta_F \mu,$$

where we abbreviate μ_X to μ, and where $M_1^F/\alpha = \mathbb{E}[s_F|\Psi_1, \Psi_0, v]$.

These will coincide with the on-path equilibrium strategies we already know:

$$\alpha_L = \hat{\alpha}_L + \frac{\hat{\nu}_L}{\alpha}$$
$$\alpha_F = \hat{\alpha}_F + \frac{\hat{\nu}_F}{\alpha}.$$

The follower's objective is

$$\sup_{\theta^F} \mathbb{E}[(v + X_0^L + \theta^L + X_0^F + \theta^F)(X_0^F + \theta^F) - (P_1 + \Lambda_2\Psi_2)\theta^F - \frac{1}{2}(X_0^F + \theta^F)^2 | X_0^F, s_F, \mathcal{F}_1, \theta^F],$$

where $\mathcal{F}_1$ is the sigma-algebra generated by (Ψ_0, Ψ_1, v). The first order condition is

$$0 = \mathbb{E}[v + X_0^L + \theta^L + X_0^F + \theta^F - P_1 - 2\Lambda_2\theta^F | X_0^F, s_F, \mathcal{F}_1, \theta^F] \quad \text{(IA.16)}$$
$$= v + \alpha s_F(1 + \alpha_L) + \delta_L \mu + X_0^F + \theta^F - P_1 - 2\Lambda_2\theta^F. \quad \text{(IA.17)}$$

Plugging in the extended strategy and matching coefficients yields

$$\hat{\alpha}_F = \frac{1}{2 + \alpha_L}\alpha_F$$
$$\hat{\nu}_F = \frac{\alpha(1 + \alpha_L)}{2 + \alpha_L}\alpha_F,$$

and indeed, on path, we have $\hat{\alpha}_F X_0^F + \hat{\nu}_F s_F = \hat{\alpha}_F X_0^F + \hat{\nu}_F X_0^F/\alpha = \alpha_F X_0^F$. Intuitively, the private state s_F informs the follower about the contribution to firm value of $(1 + \alpha_L)X_0^L$ in the leader's terminal position, while the private state X_0^F informs him about his own contribution X_0^F, and $X_0^F = X_0^L$ on path as we are assuming perfect correlation in the signals.

The leader's first-order condition is

$$0 = \mathbb{E}\left[v + X_0^L + \theta^L + X_0^F + \theta^F - (P_0 + \Lambda_1\{\Psi_1 - (\alpha_L + \delta_L)\mu\}) - \theta^L\Lambda_1 + (X_0^L + \theta^L)\Lambda_1\beta_F | X_0^L, s_L, \mathcal{F}_0, \theta^L\right] \quad \text{(IA.18)}$$
$$= v + X_0^L + \theta^L + \alpha s_L(1 + \alpha_F) + \delta_F\mu + (\beta_F - 1)\mathbb{E}\left[P_1 | X_0^L, s_L, \mathcal{F}_0, \theta^L\right] - \theta_L\Lambda_1 + (X_0^L + \theta^L)\Lambda_1\beta_F, \quad \text{(IA.19)}$$

where $\mathcal{F}_0$ is generated by (Ψ_0, v) and where

$$\mathbb{E}\left[P_1 | X_0^L, s_L, \mathcal{F}_0, \theta^L\right] = P_0 + \Lambda_1 \left\{\theta^L - (\alpha_L + \delta_L)\mu\right\}$$

Matching coefficients on X_0^L and s_L yields expressions for $\hat{\alpha}_L$ and $\hat{\nu}_L$ in terms of the already known (baseline model) equilibrium coefficients:

$$\hat{\alpha}_L = \frac{1 + \beta_F \Lambda_1}{2(1 - \beta_F)\Lambda_1 - 1}$$
$$\hat{\nu}_L = \frac{\alpha(1 + \alpha_F)}{2(1 - \beta_F)\Lambda_1 - 1},$$

and thus $\frac{\hat{\alpha}_L}{\hat{\nu}_L} = \frac{1+\beta_F\Lambda_1}{\alpha(1+\alpha_F)}$.

Outline of remaining steps to establish fixed point numerically

1. From the optimal extended strategies in the leader-follower game, we obtain the players' expected payoffs (immediately after leader-follower roles are assigned) from arbitrary histories in the pregame as quadratic functions $V_L(X_0^i, s_i, v, \mu_X), V_F(X_0^i, s_i, v, \mu_X)$, for any conjectured α, where ϕ determined by (IA.15).

2. Using these continuation payoffs, we write down trader i's maximization problem in the pre-game:

$$\sup_{\theta_i} \mathbb{E}[-P_{pre}\theta_i + (1-q)v\theta_i + \frac{q}{2}(V_L(\theta_i, s_i, v, \mu_X) + V_F(\theta_i, s_i, v, \mu_X))]$$

3. Next, we obtain a fixed point equation for α by imposing the first-order condition with respect to θ_i and then evaluate at the conjectured equilibrium strategy $\theta_i = \alpha s_i$.

4. Numerically, we show that this equation has a solution $\alpha^* > 0$; see left panel of Figure 1. Moreover, α^* homogeneous of degree 0 in $(\sigma, \sigma_v, \sigma_\epsilon)$.

Inducing Negative Correlation

Throughout this section, assume that $\rho_\epsilon = 0$, so that pre-game signals are uncorrelated conditional on v. We again reduce the problem of existence of an equilibrium to a fixed point equation in α and solve it numerically. This equilibrium generates negatively correlated positions from the perspective of the market maker conditional on the public information (Ψ_0, v).

Belief updating in the pregame

Given s_i, player i's beliefs about s_{-i} and v are as follows:[4]

- $s_{-i}|s_i \sim N\left(\frac{\sigma_v^2}{\sigma_v^2+\sigma_\epsilon^2}s_i, \frac{\sigma_\epsilon^2(\sigma_\epsilon^2+2\sigma_v^2)}{\sigma_v^2+\sigma_\epsilon^2}\right)$
- $v|s_i \sim N\left(\frac{\sigma_v^2}{\sigma_v^2+\sigma_\epsilon^2}s_i, \frac{\sigma_v^2\sigma_\epsilon^2}{\sigma_v^2+\sigma_\epsilon^2}\right)$

Let $\mu_\theta := \mathbb{E}\left[\frac{\theta_1+\theta_2}{2}|\Psi_0\right]$, $\mu_v := \mathbb{E}[v|\Psi_0]$, and $\mu_X := \mathbb{E}\left[\frac{\theta_1+\theta_2}{2}|\Psi_0, v\right]$. As $\mathbb{E}[\Psi_0] = 0$, we have

$$\mu_\theta = \frac{\text{Cov}\left(\frac{\theta_1+\theta_2}{2}, \Psi_0\right)}{\text{Var}\left(\Psi_0\right)}\Psi_0 = \frac{\alpha^2(2\sigma_v^2+\sigma_\epsilon^2)}{2\alpha^2(2\sigma_v^2+\sigma_\epsilon^2)+\sigma^2}\Psi_0 \tag{IA.20}$$

$$\mu_v = \frac{\text{Cov}\left(v, \Psi_0\right)}{\text{Var}\left(\Psi_0\right)}\Psi_0 = \frac{2\alpha\sigma_v^2}{2\alpha^2(2\sigma_v^2+\sigma_\epsilon^2)+\sigma^2}\Psi_0. \tag{IA.21}$$

As in the previous section, the MM sets price

$$P_{pre} = \mu_v + q(2+\alpha_L+\delta_L)\mu_\theta,$$

now with μ_v and μ_θ given by (IA.20)-(IA.21).

After v is publicly revealed, the MM's beliefs update as follows:

$$\mu_X := \mathbb{E}[X_0^i|\Psi_0, v] = \alpha v + \frac{\alpha^2\sigma_\epsilon^2}{2\alpha^2\sigma_\epsilon^2+\sigma^2}\Psi_0 \tag{IA.22}$$

$$\phi := \text{Var}(\theta_i|\Psi_0, v) = \alpha^2\sigma_\epsilon^2\left[1-\frac{\alpha^2\sigma_\epsilon^2}{2\alpha^2\sigma_\epsilon^2+\sigma^2}\right] = \frac{\alpha^2\sigma_\epsilon^2(\alpha^2\sigma_\epsilon^2+\sigma^2)}{2\alpha^2\sigma_\epsilon^2+\sigma^2} \tag{IA.23}$$

$$\rho := \text{Cov}(\theta_1, \theta_2|\Psi_0, v) = -\frac{\alpha^4\sigma_\epsilon^4}{2\alpha^2\sigma_\epsilon^2+\sigma^2}. \tag{IA.24}$$

Note that in the numerical solution we find, $\alpha \neq 0$, so indeed $\rho < 0$; that is, the pregame induces negatively correlated positions.

Unlike in the perfect correlation case, the players must also use Ψ_0 and v to update about the other's positions entering the pregame. Players assume each other are on path. Given $X_0^i(=\theta_i)$ and v, players' private beliefs, on and off path, entering the leader-follower game are $X_0^{-i}|X_0^i \sim N(Y_0^i, \nu_0^i)$, where

$$Y_0^i = \mathbb{E}[\alpha S^{-i}|\Psi_0, \theta^i, v] = v + \frac{\alpha\sigma_\epsilon^2}{\alpha^2\sigma_\epsilon^2+\sigma^2}(\Psi_0 - \theta^i - \alpha v)$$

$$\nu_0^i = \frac{\sigma_\epsilon^2\sigma^2}{\sigma^2+\alpha^2\sigma_\epsilon^2}.$$

[4] The posterior covariance is not needed.

While players could use v to form better estimates of each other's signals, since those signals are payoff irrelevant in the continuation game, and thus this exercise is unnecessary. (Indeed, as above, signals are not used to forecast the other's position.)

The leader-follower continuation game From the preceding discussion, players' only relevant private information in the leader-follower game is their position X_0^i. Expected payoffs will depend on v, but v does not affect the players' strategies (when the follower's strategy is written as $\theta^F = \alpha_F(X_0^F - M_1^F)$), since it is a public additive component of firm value. Let $V_L(x, \mu, v)$ and $V_F(x, \mu, v)$ (quadratic functions) denote the players' expected payoffs from the leader-follower continuation game after roles are assigned, with the information structure parameters ϕ and ρ given by (IA.22)-(IA.24) given a conjectured coefficient α.

Player i's best response problem in the pregame is now

$$\sup_{\theta_i} \mathbb{E}[-P_{pre}\theta_i + (1-q)v\theta_i + \frac{q}{2}(V_L(\theta_i, \mu_X, v) + V_F(\theta_i, \mu_X, v))].$$

The first order condition yields a fixed point equation for α, and we show numerically that it has a solution $\alpha^* > 0$, inducing negatively correlated positions; see the right panel of Figure 1. This solution is again homogeneous of degree zero in $(\sigma, \sigma_v, \sigma_\epsilon)$.

Remark 2. *If we further specialize to $\sigma_v = 0$, then players' signals are no longer payoff relevant even in the pre-game, and the model effectively induces a mixed strategy equilibrium, with the signals serving as independent randomization devices. Since signals are payoff irrelevant, the first order conditions is the same for each signal, and this means that players are indifferent over all possible trades.*

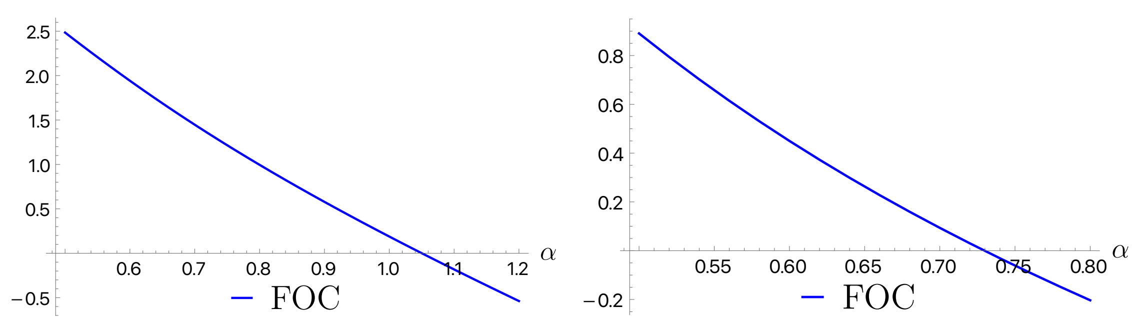

Figure 1: First order condition (FOC) in pregame evaluated at the conjectured strategy α, as a function of α. Parameter values: $\sigma = \sigma_v = \sigma_\epsilon = q = 1$, $\rho_\epsilon = 1$ (left) and $\rho_\epsilon = 0$ (right).

The left and right panels of Figure 1 show the fixed points leading to perfect positive correlation and (imperfect) negative correlation, respectively. The fixed point in the perfect positive correlation case is larger, reflecting more intense trading in the pre-game; this is consistent with the “rat race” phenomenon in dynamic trading models with correlated private information (Foster and Viswanathan, 1996).