Supplementary Appendix to "Signaling with Private Monitoring" (not for publication)

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Contents

S.1 Monetary Policy Game (Section 4.1): Proofs of Propositions 3 and 4

In this section, we prove Propositions 3 and 4. We often use NX to refer to the case $\sigma_X = +\infty$, where the X signal is pure noise, and pub to refer to the $\sigma_X = 0$ case.

We begin by proving Proposition 3(i), since only this part relates to the interior case.

S.1.1 Proof of Proposition 3 part (i)

From the proof of Proposition 2, for $\sigma_X \in (0, +\infty)$, $\dot{\beta}_0$ converges uniformly to $\frac{k\gamma^o}{4\sigma_Y^2}$ as $T \to 0$. For the public case, we borrow the β_0-ODE (to be derived later)

$$\dot{\beta}_{0t} = \beta_{3t}[2r(\beta_{0t} - k) + k(\beta_{1t} + \beta_{3t})\gamma_t/\sigma_Y^2],$$

where $\beta_{1t} + \beta_{3t} = 1$, as we also prove later. Note that this ODE can also be obtained by setting $\chi_t = 0$ and then $\sigma_X = 0$ in the β_0-ODE in the proof of Proposition 2 and replacing α_{3t}^2 attached to k with $\beta_{3t}(\beta_{1t} + \beta_{3t})$. The reason for the difference is that in the interior case, the receiver plays $\delta_1 = \alpha_3$, whereas in the public case, the receiver plays $\delta_1 = \beta_1 + \beta_3 = 1$. Following the same logic as in the interior case, this converges uniformly to $\frac{k\gamma^o}{2\sigma_Y^2}$ as $T \to 0$. (The reason the denominator carries a factor of 2 rather than 4 is again because the receiver places full weight on $\hat{M}$ in the public case, rather than splitting this weight between $\hat{M}$ and L.) Thus, there exists $T^\dagger$ such that for all $T < T^\dagger$, $\dot{\beta}_0$ is strictly higher in the public case. Since $\beta_{0T} = k$ for both cases, it follows that for sufficiently small T, β_{0t} is higher in the interior case than in the public case for all $t < T$.

Before returning to prove the rest of Proposition 3, we prove the following proposition, which subsumes Proposition 4 and establishes the properties stated in footnote 26.

Proposition S.1. *If $\sigma_X \in \{0, +\infty\}$, then an LME exists for all $T > 0$ and $r \geq 0$. Moreover,*

- *(i) If $\sigma_X = 0$, then $a_t = \beta_{0t}^{pub} + \beta_{3t}^{pub}\theta + (1 - \beta_{3t}^{pub})\hat{M}_t$ and $\hat{a}_t = \beta_{0t}^{pub} + \hat{M}_t$, where $\frac{d\beta_{3t}^{pub}}{dt} < 0$, $\beta_{3t}^{pub} \in (1/2, 1)$, and $\beta_{0t}^{pub} < k$ for $t < T$; and $(\beta_{0T}^{pub}, \beta_{3T}^{pub}) = (k, 1/2)$.*
- *(ii) If $\sigma_X = +\infty$, then $a_t = \beta_{0t}^{NX} + \alpha_{3t}^{NX}\theta + (1 - \alpha_{3t}^{NX})\mu$, where $\alpha_{3t}^{NX} \in (1/2, 1)$ for all $t \in [0, T]$, $\beta_{0t}^{NX} < k$ for $t < T$, and $\beta_{0T}^{NX} = k$. Also, $\frac{d\alpha_{3t}^{NX}}{dt} > 0$ if $r > 0$ (and constant for $r = 0$).*

If $r = 0$, then $\beta_{3,0}^{pub} > \alpha_{3,0}^{NX}$ and $\beta_{3T}^{pub} < \alpha_{3T}^{NX}$.

S.1.2 Proof of Proposition S.1: $\sigma_X = 0$ Case

The first-order condition applied to the right hand side of the HJB equation presented in Appendix B and applied at the conjectured strategy $a_t^* := \beta_{0t} + \beta_{1t} m + \beta_{3t}\theta$ reads

$$0 = -\frac{1}{2}(\beta_{0t} + \beta_{1t} m + \beta_{3t}\theta - \theta) - \frac{1}{2}[\beta_{3t}(\theta - m) - k] + (\beta_{3t}\gamma_t/\sigma_Y^2)[v_{2t} + 2m v_{4t} + \theta v_{5t}].$$

Provided $\beta_{3t}, \gamma_t > 0$ (as we verify later), $(v_{2t}, v_{4t}, v_{5t}) = \left(\frac{\sigma_Y^2(\beta_{0t}-k)}{2\beta_{3t}\gamma_t}, \frac{\sigma_Y^2(\beta_{1t}-\beta_{3t})}{4\beta_{3t}\gamma_t}, \frac{\sigma_Y^2(2\beta_{3t}-1)}{2\beta_{3t}\gamma_t}\right)$, due to the FOC holding for all $(\theta, m, t) \in \mathbb{R}^2 \times [0, T]$. And since $v_{iT} = 0$ for $i \in \{0, \ldots, 5\}$, we deduce that $(\beta_{0T}, \beta_{1T}, \beta_{3T}) = (k, 1/2, 1/2)$.

Inserting a_t^* into the HJB equation, and using the previous expressions for (v_{2t}, v_{4t}, v_{5t}) to replace $(v_{2t}, v_{4t}, v_{5t}, \dot{v}_{2t}, \dot{v}_{4t}, \dot{v}_{5t})$, yields an equation in $\vec{\beta} := (\beta_0, \beta_1, \beta_3)$ and $\dot{\vec{\beta}}$. Grouping by coefficients $(\theta, m, \theta^2$,..., etc.) in the latter yields a system of ODEs for $(v_0, v_1, v_3, \beta_0, \beta_1, \beta_3)$:

$$\dot{\beta}_{0t} = \beta_{3t}\left[2r(\beta_{0t} - k) + \frac{k\gamma_t(\beta_{1t} + \beta_{3t})}{\sigma_Y^2}\right] \tag{S.1}$$

$$\dot{\beta}_{1t} = \beta_{3t}\left[r(2\beta_{1t} - 1) + \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2}\right]$$

$$\dot{\beta}_{3t} = \beta_{3t}\left[r(2\beta_{3t} - 1) - \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2}\right]$$

along with $\dot{v}_{0t} = \frac{k^2}{4} + r v_{0t} + \frac{\beta_{0t}^2}{4} + \frac{\beta_{3t}}{4}\gamma_t(\beta_{3t} - \beta_{1t})$, $\dot{v}_{1t} = r v_{1t} - \frac{\beta_{0t}}{2}$, and $\dot{v}_{3t} = \frac{1}{4} + r v_{3t} - \frac{\beta_{3t}^2}{2}$, with terminal conditions $(v_{0T}, v_{1T}, v_{3T}, \beta_{0T}, \beta_{1T}, \beta_{3T}) = (0, 0, 0, k, 1/2, 1/2)$, coupled with $\dot{\gamma}_t = -\beta_{3t}^2\gamma_t^2/\sigma_Y^2$ and initial condition $\gamma_0 = \gamma^o$. After we solve the subsystem $(\beta_1, \beta_3, \gamma)$, one easily obtains β_0 from its ODE and then (v_0, v_1, v_3), as the latter ODEs are uncoupled from one another and linear in themselves.

We now solve the BVP in $(\beta_1, \beta_3, \gamma)$ using a backward IVP: abusing notation,

$$\dot{\beta}_{1t} = -\beta_{3t}\left[r(2\beta_{1t} - 1) + \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2}\right] \tag{S.2}$$

$$\dot{\beta}_{3t} = -\beta_{3t}\left[r(2\beta_{3t} - 1) - \frac{\beta_{1t}\beta_{3t}\gamma_t}{\sigma_Y^2}\right] \tag{S.3}$$

$$\dot{\gamma}_t = \left(\frac{\beta_{3t}\gamma_t}{\sigma_Y}\right)^2 \tag{S.4}$$

with initial conditions $\beta_{1,0} = \beta_{3,0} = \frac{1}{2}$ and $\gamma_0 = \gamma^F \geq 0$.

We shall argue via the intermediate value theorem that there exists γ^F such that $\gamma_T = \gamma^o$, thus solving the BVP. To that end, we make use of the following lemma, which establishes uniform bounds and other properties for the equilibrium coefficients. Define $B_t^{\text{pub}} := \beta_{1t} + \beta_{3t}$.

Lemma S.1. *Fix any $\gamma^F \geq 0$. If a solution to the backward system exists over $[0,T]$, then any such solution must have the following properties. If $\gamma^F > 0$, then (i) $B_t^{pub} = 1$ for all $t \in [0,T]$, (ii) $\beta_{3t} \in (1/2, 1)$ and $\beta_{1t} \in (0, 1/2)$ for all $t \in (0,T]$, (iii) $\dot{\beta}_3 > 0$ is while $\dot{\beta}_1 < 0$, and (iv) γ is strictly increasing. If $\gamma^F = 0$, then $\beta_{1t} = \beta_{3t} = \frac{1}{2}$ and $\gamma_t = 0$ for all $t \in [0,T]$.*

Proof of Lemma S.1. Because the system (S.2)-(S.4) is C^1, the solution is unique when it exists. If $\gamma^F = 0$, it is clear by inspection that $(\beta_1, \beta_3, \gamma) = (1/2, 1/2, 0)$ (uniquely) solves the IVP, so assume hereafter that $\gamma^F > 0$. We first claim that $\beta_3 > 0$. Indeed, let $f^{\beta_3}(t, \beta_{3t})$ denote the RHS of the β_3-ODE in (S.3). Letting $x_t := 0$ for all $t \in [0,T]$, we have $\beta_{3,0} = 1/2 > x_0$ and $\dot{\beta}_{3t} - f^{\beta_3}(t, \beta_{3t}) = 0 = \dot{x}_t - f^{\beta_3}(t, x_t)$; by the comparison theorem, the claim follows. Now, add the ODEs that β_1 and β_3 satisfy to get $\dot{B}_t^{\text{pub}} = 2r\beta_{3t}(1 - B_t^{\text{pub}})$ with $B_0^{\text{pub}} = 1$; because the RHS is of class C^1, it has a unique solution, which is clearly $B^{\text{pub}} = 1$. Hence, $\beta_1 + \beta_3 = 1$ and $\dot{\beta}_{3t} = \beta_{3t}\left[r(1 - 2\beta_{3t}) + \frac{\beta_{3t}(1-\beta_{3t})\gamma_t}{\sigma_Y^2}\right]$, and we maintain the label $f^{\beta_3}(t, \beta_{3t})$ for its RHS. Defining $x_t := 1$ for all $t \in [0,T]$, then, $x_0 = 1 > \beta_{3,0} = \frac{1}{2}$, and $\dot{\beta}_{3t} - f^{\beta_3}(t, \beta_{3t}) = 0 \leq r = \dot{x}_t - f^{\beta_3}(t, x_t)$; thus, $\beta_3 < 1$ and $\beta_1 = 1 - \beta_3 > 0$.

Since $\beta_3 > 0$, γ is clearly strictly increasing, and hence $\gamma_t > 0$ for all $t \in [0,T]$. Now, $\dot{\beta}_{3t} = \frac{1}{2}\left[0 + \frac{\gamma_t}{4\sigma_Y^2}\right] > 0$ whenever $\beta_{3t} = \frac{1}{2}$, and thus $\beta_{3t} > 1/2$ and $\beta_{1t} < 1/2$ for all $t \in (0,T]$.

We now turn to (iii). Since $\dot{\beta}_{1t} + \dot{\beta}_{3t} = 0$, we just show that $\dot{\beta}_3 > 0$; in turn, it suffices to show that $H_t := \dot{\beta}_{3t}/\beta_{3t} = r(1 - 2\beta_{3t}) + \frac{\beta_{3t}(1-\beta_{3t})\gamma_t}{\sigma_Y^2} > 0$ for all $t \in [0,T]$. For $t = 0$, $H_0 = \frac{\gamma_0}{4\sigma_Y^2} > 0$ immediate from inspection. For $t > 0$, by solving $H_t = 0$ for r (which is valid as $\beta_{3t} \neq 1/2$ for $t > 0$), whenever $H_t = 0$, it must be that $\dot{H}_t = \frac{(1-\beta_{3t})\beta_{3t}^3\gamma_t^2}{\sigma_Y^4} > 0$. It follows that $H_t > 0$ for all t as desired. □

Given the uniform bounds established in Lemma S.1, we solve the BVP through a *shooting* step, arguing by contradiction as in Bonatti et al. (2017). Note that if $\gamma^F = 0$, the IVP has the (unique) static solution. Define

$$\bar{\gamma} := \sup\{\tilde{\gamma}^F > 0 \mid \text{a solution to the IVP exists over } [0,T] \text{ for all } \gamma^F \in (0, \tilde{\gamma}^F)\}.$$

Since the right-hand side of the equations that comprise the IVP are of class C^1, the solution is unique when it exists, and there is continuous dependence of the solution on the initial conditions; in particular, the terminal value γ_T is continuous in γ^F (see Theorem on page 397 in Hirsch et al. (2004)). Hence if there exists $\gamma^F \in (0, \bar{\gamma})$ such that $\gamma_T(\gamma^F) \geq \gamma^o$, by the intermediate value theorem there exists a $\gamma^F \in (0, \bar{\gamma})$ such that $\gamma_T(\gamma^F) = \gamma^o$, allowing us to construct a solution to the BVP.

Suppose then that for all $\gamma^F \in (0, \bar{\gamma})$, $\gamma_T(\gamma^F) < \gamma^o$. In particular, because γ_t is non-decreasing in the backward system for any initial condition, we have that $\gamma_t \in (0, \gamma^o)$ does not explode and the uniform bounds from the lemma apply. We first claim that a solution to the IVP for $\gamma^F = \bar{\gamma}$ must exist over $[0, T]$. To see this, let $[0, \tilde{T})$ denote the maximal interval of existence, and suppose by way of contradiction that $\tilde{T} \in (0, T]$. Thus, there must be some function $x(\cdot, \bar{\gamma})$ which explodes at $\tilde{T}$, and so, for $\tilde{t} \in (0, \tilde{T})$ sufficiently close to $\tilde{T}$, we have $x(\tilde{t}, \bar{\gamma}) \notin [0, 1]$. But for any sequence $(\gamma_n^F)_{n\in\mathbb{N}}$ taking values in $(0, \bar{\gamma})$ such that $\gamma_n^F \uparrow \bar{\gamma}$, by continuity of solutions with respect to initial conditions, we have $x(\tilde{t}, \bar{\gamma}) = \lim_{n\to\infty} x(\tilde{t}, \gamma_n^F) \in [0, 1]$, a contradiction. We conclude that a solution to the IVP for $\gamma^F = \bar{\gamma}$ must exist over $[0, T]$, and hence, by the extensibility of the solutions (Theorem on page 397 in Hirsch et al. (2004)), that a solution must also exist for all $\gamma^F \in [\bar{\gamma}, \bar{\gamma} + \epsilon)$, some $\epsilon > 0$, thereby violating the definition of $\bar{\gamma}$ as a supremum.

Thus, a solution to the BVP exists. Moreover, Lemma S.1 establishes (for reversed time) the properties of β_3 stated in Proposition S.1. Since $\beta_{1t} + \beta_{3t} = 1$, we have $a_t = \beta_{0t} + \beta_{3t}\theta + (1 - \beta_{3t})\hat{M}_t$, which implies $\hat{a}_t = \hat{\mathbb{E}}_t[a_t] = \beta_{0t} + \hat{M}_t$. As $\beta_3 \geq 1/2$ is finite, we have $\beta_3, \gamma > 0$ allowing us to recover (v_2, v_4, v_5) through the identities stated earlier, and then (β_0, v_0, v_1, v_3) are pinned down as argued above. As for the claims about β_0, we have $\beta_{0T} = k$, and from (S.1), using that $\beta_{1t} + \beta_{3t} = 1$, whenever $\beta_{0t} = k$ we have $\dot{\beta}_{0t} = \frac{k\beta_{3t}\gamma}{\sigma_Y^2} > 0$, and thus β_{0t} must lie below k until time T.

S.1.3 Proof of Proposition S.1: $\sigma_X = +\infty$ Case

Proof of Lemma B.1. Anticipating $a_t = \alpha_{0t} + \alpha_{2t}\mu + \alpha_{3t}\theta$, the receiver's belief is $\sim \mathcal{N}(\hat{M}_t, \gamma_t)$ where $d\hat{M}_t = \frac{\alpha_{3t}\gamma_t}{\sigma_Y^2}[dY_t - (\alpha_{0t} + \alpha_{2t}\mu + \alpha_{3t}\hat{M}_t)dt]$ and $\dot{\gamma}_t = -\frac{\gamma_t^2\alpha_{3t}^2}{\sigma_Y^2}$. Thus, $\hat{M}_t = \mu R(t, 0) + \int_0^t R(t, s)\frac{\alpha_{3s}\gamma_s}{\sigma_Y^2}[(a_s - \alpha_{0s} - \alpha_{2s}\mu)ds + \sigma_Y dZ_s^Y]$ and $M_t = \mu R(t, 0) + \int_0^t R(t, s)\frac{\alpha_{3s}\gamma_s}{\sigma_Y^2}(a_s - \alpha_{0s} - \alpha_{2s}\mu)ds$ where $R(t, s) = \exp(-\int_s^t \frac{\alpha_{3u}^2\gamma_u}{\sigma_Y^2}du)$. Solving for M after inserting $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{2t}\mu + \beta_{3t}\theta$, and imposing the representation, it is easy to conclude that $M_t = \chi_t\theta + (1 - \chi_t)\mu$ will hold if and only if $\dot{\chi}_t = \frac{\alpha_{3t}^2\gamma_t}{\sigma_Y^2}(1 - \chi_t)$. By arguments analogous to those used for Lemma A.1, the (γ, χ)-ODE pair admits a unique solution, and it satisfies $\chi = 1 - \gamma/\gamma^o$. □

As noted in Appendix B, fixing μ, (θ, M_t, t) are the relevant states for the sender. In more detail, the receiver expects that the sender is always on path and therefore to be playing $a_t = \alpha_{0t} + \alpha_{2t}\mu + \alpha_{3t}$ by the representation. The receiver's best response is thus $\hat{a}_t = \hat{\mathbb{E}}_t[a_t] = \alpha_{0t} + \alpha_{2t}\mu + \alpha_{3t}\hat{M}_t$. Taking an expectation of the sender's flow payoff $\frac{1}{4}[-(a_t - \theta)^2 - (a_t - \hat{a}_t - k)^2]$ then yields that (θ, M_t, t) is the relevant state on and off path. (Indeed, expanding the squares in the previous expression, the only nontrivial component is $\mathbb{E}_t[\hat{a}_t^2]$, which makes $\mathbb{E}_t[\hat{M}_t^2]$ appear; however, $\mathbb{E}_t[\hat{M}_t^2] = M_t^2 + \mathbb{E}_t[(\hat{M}_t - M_t)^2] = M_t^2 + \gamma_t\chi_t$ at all histories.[1])

[1]From the proof of Lemma B.1, $\mathbb{E}_t[(\hat{M}_t - M_t)^2] = \mathbb{E}_t[(\int_0^t R(t, s)\frac{\alpha_{3s}\gamma_s}{\sigma_Y}dZ_s^Y)^2] = \int_0^t R(t, s)^2\frac{\alpha_{3s}^2\gamma_s^2}{\sigma_Y^2}ds =$

The HJB equation and the law of motion for γ yields a core BVP consisting of $(\beta_1, \beta_2, \beta_3, \gamma)$. Using the same method used for the $\sigma_X = 0$ case, we construct a backward IVP version of our original BVP that has a parametrized initial condition γ^F for the $\gamma-$ODE:

$$\begin{aligned}
\dot{\beta}_{1t} &= \alpha_{3t}(2\sigma_Y^2)^{-1} \times \left\{ r\sigma_Y^2 - 2\beta_{1t}[\beta_{3t}\gamma_t + r\sigma_Y^2(2-\chi_t)] + 2\beta_{1t}^2\gamma_t(1-\chi_t) \right\} \\
\dot{\beta}_{2t} &= \alpha_{3t}(2\sigma_Y^2)^{-1} \times \left\{ -2r\sigma_Y^2\beta_{2t}(2-\chi_t) + r\sigma_Y^2(1-\chi_t) - 2\gamma_t\beta_{1t}^2(1-\chi_t) \right\} \\
\dot{\beta}_{3t} &= \alpha_{3t}(2\sigma_Y^2)^{-1} \times \left\{ r\sigma_Y^2(2-\chi_t) + 2\beta_{3t}[\beta_{1t}\gamma_t - r\sigma_Y^2(2-\chi_t)] \right\} \\
\dot{\gamma}_t &= \alpha_{3t}^2\gamma_t^2/\sigma_Y^2
\end{aligned}$$

with initial condition $(\beta_{1,0}, \beta_{2,0}, \beta_{3,0}, \gamma_0) = (\frac{1}{2(2-\chi_0)}, \frac{1-\chi_0}{2(2-\chi_0)}, \frac{1}{2}, \gamma^F)$ and where $\chi = 1 - \gamma/\gamma^o$.

We aim to prove that there exists $\gamma^F \in (0, \gamma^o)$ such that the IVP has a (unique) solution which satisfies $\gamma_T = \gamma^o$. ($\gamma^F = 0$ cannot work, as $(\beta_1, \beta_2, \beta_3, \gamma) = (1/2, 0, 1/2, 0)$ is the unique solution.) As argued in the proof of the $\sigma_X = 0$ case, it suffices to show that the system is uniformly bounded if $\gamma_t \in [0, \gamma^o]$ over $[0, T]$.

The α_3-ODE is $\dot{\alpha}_{3t} = f^\alpha(t, \alpha_{3t}) := r\alpha_{3t}[1 - \alpha_{3t}(2-\chi_t)]$ and $\alpha_{3,0} = \frac{1}{2-\chi_0} > 0$. By the comparison theorem, $\alpha_3 > 0$; hence, by the same argument as in the proof of Lemma S.1, γ is increasing (in the backward system), so $\chi = 1 - \gamma/\gamma^o < 1$ is decreasing. As $\alpha_{3,0} = \frac{1}{2-\chi_0}$ and $\dot{\alpha}_{3,0} > \frac{d}{dt}\left(\frac{1}{2-\chi_t}\right)|_{t=0}$, the comparison theorem can be applied to α_3 and $1/(2-\chi)$ to show $\alpha_{3t} \geq 1/(2-\chi_t) \geq 1/2$, with both inequalities strict for all $t \in (0, T]$, for all $r \geq 0$. And $\alpha_{3t} \geq 1/(2-\chi_t)$ implies $\dot{\alpha_{3t}} \leq 0$ (and hence $\dot{\alpha}_{3t} \geq 0$ in the forward system) for all $t \in [0, T]$, with strict inequality for $t \in (0, T]$ if and only if $r > 0$; for $r = 0$, α_3 is constant. It follows that for all $t \in (0, T]$, $\alpha_{3t} \leq \alpha_{3,0} = \frac{1}{2-\chi_0} < 1$.

Now, $B^{\text{NX}} := \beta_1 + \beta_2 + \beta_3$ satisfies $\dot{B}_t^{\text{NX}} = r\alpha_{3t}(2-\chi_t)(1 - B_t^{\text{NX}})$ with $B_0^{\text{NX}} = 1$; thus $B^{\text{NX}} \equiv 1$. This establishes that in any LME, $a_t = \beta_{0t} + (1-\alpha_{3t})\mu + \alpha_{3t}\theta$.

Next, we establish uniform bounds on β_1 and β_3 (and hence β_2). Toward showing $\beta_1 > 0$, observe that the RHS of the β_1-ODE can be written as $f^{\beta_1}(t, \beta_1)$ of class C^1. Letting $x := 0$, we have $\beta_{10} > x_0 = 0$ and $\dot{x}_t - f^{\beta_1}(t, x_t) = 0 - \frac{\alpha_{3t}}{2\sigma_Y^2}r\sigma_Y^2 \leq 0 = \dot{\beta}_{1t} - f^{\beta_1}(t, \beta_{1t})$ and thus by the comparison theorem, $\beta_1 > x = 0$. This implies that $\beta_3 = \alpha_3 - \beta_1\chi \leq \alpha_3 < 1$. We now show $\beta_{3t} > 1/2$ and $\beta_{1t} < \beta_{1t}^m := \frac{1}{2(2-\chi_t)} < 1$ for all $t \in (0, T]$. For the former, recall that $\beta_{30} = 1/2$, and whenever $\beta_{3t} = 1/2$, $\dot{\beta}_{3t} = \frac{\alpha_{3t}\beta_{1t}\gamma_t}{2\sigma_Y^2} > 0$; it follows that $\beta_{3t} > 1/2$ for all $t \in (0, T]$. Now $\beta_{10} = \beta_{10}^m < 1$, and for all $t \in [0, T)$ (where $\chi_t > 0$),

$$\dot{\beta}_{1t}^m - f^{\beta_1}(t, \beta_{1t}^m) = \frac{\gamma_t\left(\beta_{3t}[2-\chi_t] - [1-\chi_t]\right)\left(2\beta_{3t}[2-\chi_t] + \chi_t\right)}{4\sigma_Y^2(2-\chi_t)^4} > 0 = \dot{\beta}_{1t} - f^{\beta_1}(t, \beta_{1t}),$$

$\int_0^t \exp(2\int_s^t \frac{\dot{\gamma}_u}{\gamma_u}du)(-\dot{\gamma}_s)ds = \int_0^t (\gamma_t/\gamma_s)^2(-\dot{\gamma}_s)ds = \gamma_t^2(1/\gamma_t - 1/\gamma^o) = \gamma_t\chi_t$.

from which $\dot{\beta}^m_{10} > \dot{\beta}_{10}$. By the comparison theorem, $\beta_{1t} < \beta^m_{1t} < 1$ for all $t \in (0, T]$.

Via the identity $B^{\mathrm{NX}} \equiv 1$, the uniform bounds just established imply uniform bounds on β_2. Thus, by the same one-dimensional shooting argument used for the $\sigma_X = 0$ case, a solution to the BVP for $(\beta_1, \beta_2, \beta_3, \gamma)$ exists.

Going forward in time once again, β_0 is uniquely determined by the terminal condition $\beta_{0T} = k$ and ODE $\dot{\beta}_{0t} = \alpha_{3t}\left[(2-\chi_t)r(\beta_{0t}-k) + \frac{k\alpha_{3t}\gamma_t}{\sigma_Y^2}\right]$ which is linear in β_0. Since the right hand side reduces to $\frac{k\alpha_{3t}^2\gamma_t}{\sigma_Y^2}$ whenever $\beta_{0t} = k$, we have $\beta_{0t} < k$ for all $t \in [0, T)$.

Now $\alpha_3 > 0$, and further, $\gamma > 0$ since α_3 is finite. Hence, from $(\beta_0, \beta_1, \beta_2, \beta_3)$, the coefficients (v_2, v_5, v_7, v_9) are backed out directly as in the proof of Theorem 1. The ODEs for the remaining value function coefficients are linear and uncoupled, so they also have unique solutions. Thus, we have solved the HJB equation and characterized an LME.

S.1.4 Proof of Proposition S.1: Comparison of Signaling Coefficients

The ranking of signaling coefficients at time T is immediate: we have $\beta^{\mathrm{pub}}_{3T} = 1/2 < 1/(2-\chi_T) = \alpha^{\mathrm{NX}}_{3T}$ (independent of the discount rate). To establish the ranking at time 0 for $r = 0$, we use two lemmata that provide closed-form solutions.

Lemma S.2 (Closed-form solution: public case, $r = 0$)**.** *For $r = 0$, the monetary policy game has a unique LME for the public case, and $(\beta_0, \beta_1, \beta_3, \gamma)$ satisfy*

$$\beta_{0t} = k\left[1 - \ln\left(\frac{2\sigma_Y^2}{2\sigma_Y^2 - \gamma_T(T-t)}\right)\right], \beta_1 \equiv 1 - \beta_3$$

$$\gamma_t = \frac{\gamma_T}{2} + \frac{1}{\frac{2}{\gamma_T} - \frac{T-t}{\sigma_Y^2}}, \beta_{3t} = \frac{1}{2 - \frac{\gamma_T(T-t)}{2\sigma_Y^2}}, \text{ and } \gamma_T = \frac{\gamma^o T + 2\sigma_Y^2 - \sqrt{(\gamma^o T)^2 + 4\sigma_Y^4}}{T}.$$

Proof. Observe that $\dot{\beta}_{3t}\gamma_t + \beta_{3t}\dot{\gamma}_t = \frac{\beta_{3t}^2\gamma_t^2}{\sigma_Y^2}$. Hence, define $\Pi_t := \beta_{3t}\gamma_t$, which has ODE $\dot{\Pi}_t = \frac{\Pi_t^2}{\sigma_Y^2}$ with initial condition $\Pi_0 = \beta_{3,0}\gamma^{F,Pub} = \gamma^{F,Pub}/2$, where the variable $\gamma^{F,Pub}$ denotes our auxiliary parameter γ^F introduced earlier, for this special public case. The solution to this ODE is $\Pi_t = \left[\frac{2}{\gamma^{F,\mathrm{pub}}} - \frac{t}{\sigma_Y^2}\right]^{-1}$. Substitute Π into $\dot{\gamma}_t = -\frac{\beta_{3t}^2\gamma_t^2}{\sigma_Y^2}$ to obtain $\dot{\gamma}_t = \frac{1}{\sigma_Y^2}\left[\frac{2}{\gamma^{F,\mathrm{pub}}} - \frac{t}{\sigma_Y^2}\right]^{-2}$ which implies $\gamma_t = C_\gamma + \left[\frac{2}{\gamma^{F,\mathrm{pub}}} - \frac{t}{\sigma_Y^2}\right]^{-1}$. As $\gamma_0 = \gamma^{F,\mathrm{pub}}$, we have $C_\gamma = \gamma^{F,\mathrm{pub}}/2$ and thus

$$\gamma_t = \frac{\gamma^{F,\mathrm{pub}}}{2} + \left[\frac{2}{\gamma^{F,\mathrm{pub}}} - \frac{t}{\sigma_Y^2}\right]^{-1}. \quad \text{(S.5)}$$

Moreover, $\gamma_T = \gamma^o = \frac{\gamma^{F,\text{pub}}}{2} + \left[\frac{2}{\gamma^{F,\text{pub}}} - \frac{T}{\sigma_Y^2}\right]^{-1}$, which is equivalent to the quadratic $\frac{T}{2}\left(\gamma^{F,\text{pub}}\right)^2 - \left(\gamma^o T + 2\sigma_Y^2\right)\gamma^{F,\text{pub}} + 2\sigma_Y^2\gamma^o = 0$. The quadratic on the LHS is convex and evaluates to $2\sigma_Y^2\gamma^o > 0$ at $\gamma^{F,\text{pub}} = 0$ and evaluates to $-\left(\gamma^o\right)^2 T/2 < 0$ at $\gamma^{F,\text{pub}} = \gamma^o$, so there is a unique solution in $(0, \gamma^o)$ which in the forward system is γ_T as in the proposition statement. Substituting this into (S.5) and returning to the forward system by replacing t with $T - t$ yields γ_t in the forward system. It is easy to verify that $\gamma_t > 0$ for all t.

We now characterize $(\beta_0, \beta_1, \beta_3)$. In the forward system, $\beta_{3t} = \Pi_t/\gamma_t = [2 - \frac{\gamma_T^{\text{pub}}(T-t)}{2\sigma_Y^2}]^{-1}$ and $\beta_{1t} = 1 - \beta_{3t}$. Finally, using $r = 0$, the β_0-ODE reduces to $\dot{\beta}_{0t} = k\beta_{3t}\gamma_t/\sigma_Y^2$. Writing $\beta_{0t} = k - \int_t^T \dot{\beta}_{0s}\, ds$, using the expressions above for β_3 and γ, and carrying out the integration yields the stated solution for β_0. □

Lemma S.3 (Closed-form solution: $\sigma_X = +\infty$ case, $r = 0$)**.** *For $r = 0$, the coordination game has a unique LME for the $\sigma_X = +\infty$ case:*

$$\beta_{1t} = \frac{\gamma^o[(\gamma^o + \gamma_T)^2\sigma_Y^2 - (T-t)(\gamma^o)^2\gamma_T]}{(\gamma^o + \gamma_T)[2\sigma_Y^2(\gamma^o + \gamma_T)^2 - (T-t)(\gamma^o)^2\gamma_T]}, \quad \beta_{3t} = \frac{\sigma_Y^2(\gamma^o + \gamma_T)^2}{2\sigma_Y^2(\gamma^o + \gamma_T)^2 - (T-t)(\gamma^o)^2\gamma_T}$$

$$\beta_{0t} = k\left[1 - \ln\left(\gamma_t/\gamma_T\right)\right], \ \alpha_{3t} = \frac{\gamma^o}{\gamma^o + \gamma_T}, \qquad \gamma_t = \frac{\gamma_T\sigma_Y^2\left(\gamma^o + \gamma_T\right)^2}{\sigma_Y^2\left(\gamma^o + \gamma_T\right)^2 - (T-t)\left(\gamma^o\right)^2\gamma_T},$$

for all $t \in [0, T]$, where $\chi_t = 1 - \gamma_t/\gamma^o$ and $\gamma_T \in (0, \gamma^o)$ is the unique solution in $(0, \gamma^o)$ to the cubic $q(\gamma) := \gamma T\left(\gamma^o\right)^3 + \left(\gamma - \gamma^o\right)\left(\gamma + \gamma^o\right)^2\sigma_Y^2 = 0$, and $\beta_2 \equiv 1 - \beta_1 - \beta_3$.

Proof. We work with the backward system, where the α_3-ODE is $\dot{\alpha}_{3t} = r\alpha_{3t}[1 - \alpha_{3t}(2 - \chi_t)]$. With $r = 0$, α_3 must be constant and equal to its initial value $\alpha_{3,0} = \frac{1}{2-\chi_0}$. Next, recall that by Lemma B.1, $\chi_t = 1 - \frac{\gamma_t}{\gamma^o}$, so $\chi_0 = 1 - \frac{\gamma^{F,\text{NX}}}{\gamma^o}$ and thus $\alpha_{3t} = \alpha_3 = \frac{\gamma^o}{\gamma^{F,\text{NX}} + \gamma^o}$ for all $t \in [0, T]$. (The variable $\gamma^{F,\text{NX}}$ now plays the role of $\gamma^{F,Pub}$ in the public case.) Note that the ODE $\dot{\gamma}_t = \frac{\alpha_3^2\gamma_t^2}{\sigma_Y^2}$ given an initial value $\gamma^{F,\text{NX}}$ has solution $\gamma_t = \frac{\gamma^{F,\text{NX}}\sigma_Y^2}{\sigma_Y^2 - \gamma^{F,\text{NX}}\left(\frac{\gamma^o}{\gamma^{F,\text{NX}} + \gamma^o}\right)^2 t}$; switching back to the forward system by replacing t with $T - t$ yields the expression in the original statement. The terminal condition $\gamma_T = \gamma^o$ is equivalent to a cubic equation for $\gamma^{F,\text{NX}}$:

$$q(\gamma^{F,\text{NX}}) := \gamma^{F,\text{NX}}T\left(\gamma^o\right)^3 + \left(\gamma^{F,\text{NX}} - \gamma^o\right)\left(\gamma^{F,\text{NX}} + \gamma^o\right)^2\sigma_Y^2 = 0.$$

Note $q\left(\gamma^{F,\text{NX}}\right) > 0$ for $\gamma^{F,\text{NX}} \geq \gamma^o$ and $q\left(\gamma^{F,\text{NX}}\right) \leq 0$ for $\gamma^{F,\text{NX}} \leq 0$, so all real roots must lie in $(0, \gamma^o)$. Now any root to the cubic must satisfy

$$\frac{T(\gamma^o)^3}{\gamma^o - \gamma^{F,\text{NX}}} = \sigma_Y^2\frac{(\gamma^{F,\text{NX}} + \gamma^o)^2}{\gamma^{F,\text{NX}}}. \tag{S.6}$$

The LHS of (S.6) is strictly increasing for $\gamma^{F,\text{NX}} \in (0, \gamma^o)$ while the RHS is strictly decreasing in this interval, so q has a unique real root. Returning to the β_1 ODE, using $\alpha_3 = \beta_1\chi + \beta_3$, we have $\dot{\beta}_1 = -\frac{\alpha_3\gamma_t\beta_{1t}}{\sigma_Y^2}(\alpha_3 - \beta_{1t})$ in the backward system. This ODE can be solved by integration after moving $\beta_1(\alpha_3 - \beta_1)$ to the LHS, and with algebra, one obtains (in the forward system) the expression in the proposition. One then obtains β_{3t} from these using $\beta_{3t} = \alpha_3 - \beta_{1t}\chi_t$. Finally, one calculates β_0 as $\beta_{0t} = k - \int_t^T \dot{\beta}_{0s}\, ds$ as in the public case, and the identity $\beta_2 \equiv 1 - \beta_1 - \beta_3$ was already established in the existence part of the proposition. □

Equipped with the previous two lemmata, we now prove the claim $\beta_{3,0}^{\text{pub}} > \alpha_{3,0}^{\text{NX}}$. Using the associated expressions from Lemmata S.2 and S.3, this is equivalent to

$$\frac{1}{2 - \frac{\gamma_T^{\text{pub}} T}{2\sigma_Y^2}} > \frac{\gamma^o}{\gamma^o + \gamma_T^{\text{NX}}} \iff \hat{\gamma} := \gamma^o \left(1 - \frac{\gamma_T^{\text{pub}} T}{2\sigma_Y^2}\right) < \gamma_T^{\text{NX}}.$$

Recalling the cubic equation that implicitly defines γ_T^{NX} in the proof of Lemma S.3, where q crosses 0 from below, it suffices to show that $q(\hat{\gamma}) = T\hat{\gamma}(\gamma^o)^3 + (\hat{\gamma} - \gamma^o)(\hat{\gamma} + \gamma^o)^2\sigma_Y^2 < 0$. Using the expression for γ_T^{pub} from Lemma S.2, one can show that

$$q(\hat{\gamma}) = -\frac{T(\gamma^o)^4}{2\sigma_Y^4}\left[(T\gamma^o)^2 + 2\sigma_Y^4 - T\gamma^o\sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right].$$

The expression in square brackets can be written as $\frac{x+y}{2} - \sqrt{xy} > 0$ where $x = (T\gamma^o)^2 > 0$ and $y = (T\gamma^o)^2 + 4\sigma_Y^4 > 0$, and thus $q(\hat{\gamma}) < 0$, concluding the proof that $\beta_{3,0}^{\text{pub}} > \alpha_{3,0}^{\text{NX}}$.

S.1.5 Proof of Proposition 3 part (ii)

We now turn to the comparison of β_0 coefficients for the $\sigma_X = 0$ and $\sigma_X = +\infty$ cases. Let superscript pub and NX denote the $\sigma_X = 0$ and $\sigma_X = +\infty$ cases, respectively. We make use of the following lemma, which says that more information is transmitted to the receiver when $\sigma_X = +\infty$ than when $\sigma_X = 0$.

Lemma S.4. *Fix $r = 0$ and $\gamma^o, \sigma_Y > 0$. Then for all T, $\gamma_T^{pub} > \gamma_T^{NX}$.*

Proof. Recall that γ_T^{NX} is the unique positive root of the cubic equation $q(\gamma) = 0$ defined in Lemma S.3. At γ_T^{NX}, it is easy to deduce that q must cross 0 from below, and hence to prove the claim, it suffices to show that $q(\gamma_T^{\text{pub}}) > 0$. By direct calculation,

$$q(\gamma_T^{\text{pub}}) = +\frac{\sigma_Y^2}{T^3}\left(2\sigma_Y^2 - \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right)\left(2T\gamma^o + 2\sigma_Y^2 - \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right)^2$$
$$+ (\gamma^o)^3\left(T\gamma^o + 2\sigma_Y^2 - \sqrt{(T\gamma^o)^2 + 4\sigma_Y^4}\right) = (\gamma^o)^4 T q_2(S), \qquad \text{where}$$

$$q_2(S) := 1 + 2S - \sqrt{1+4S^2} + S\left(2S - \sqrt{1+4S^2}\right)\left(2 + 2S - \sqrt{1+4S^2}\right)^2 \text{ and } S := \frac{\sigma_Y^2}{T\gamma^o}.$$

We now show that $q_2(S) > 0$ for all $S > 0$ (observe that $q_2(0) = 0$). Let $R(S) = 1 + 2S - \sqrt{1+4S^2}$; it is straightforward to verify that $R(0) = 0$ and that for all $S \geq 0$, $R'(S) > 0$ and $R(S) < 1$. Moreover, the inverse of R is the function $S : [0,1) \to [0,\infty)$ characterized by $S(R) := \frac{R(2-R)}{4(1-R)}$. Hence, by change of variables, $q_2(S) > 0$ for all $S > 0$ iff $q_3(R) > 0$, where $q_3(R) := R - S(R)(1-R)(R+1)^2$. Now for $R \in [0,1)$, $q_3(R) > 0$ if and only if $S(R) = \frac{R(2-R)}{4(1-R)} < \frac{R}{(1-R)(R+1)^2}$, if and only $q_4(R) := (2-R)(R+1)^2 < 4$. It is straightforward to verify that over the interval $[0,1]$, $q_4(R)$ attains its maximum value of 4 at $R = 1$, and tracing our steps backwards this implies that $q(\gamma_T^{\text{pub}}) > 0$. □

Recall from Lemmata S.2 and S.3 the closed-form expressions

$$\beta_{0t}^{\text{pub}} = k\left[1 - \ln\left(\frac{2\sigma_Y^2}{2\sigma_Y^2 - \gamma_T^{\text{pub}}(T-t)}\right)\right] \tag{S.7}$$

$$\beta_{0t}^{\text{NX}} = k\left[1 - \ln\left(\gamma_t^{\text{NX}}/\gamma_T^{\text{NX}}\right)\right], \tag{S.8}$$

where

$$\gamma_T^{\text{pub}} = \frac{\gamma^o T + 2\sigma_Y^2 - \sqrt{(\gamma^o T)^2 + 4\sigma_Y^4}}{T} \tag{S.9}$$

$$\gamma_t^{\text{NX}} = \frac{\gamma_T^{\text{NX}}\sigma_Y^2\left(\gamma^o + \gamma_T^{\text{NX}}\right)^2}{\sigma_Y^2\left(\gamma^o + \gamma_T^{\text{NX}}\right)^2 - (T-t)\left(\gamma^o\right)^2\gamma_T^{\text{NX}}}, \tag{S.10}$$

and where $\gamma_T^{\text{NX}} \in (0,\gamma^o)$ is the unique root in $(0,\gamma^o)$ to the cubic $q(\gamma) := \gamma T\left(\gamma^o\right)^3 + \left(\gamma - \gamma^o\right)\left(\gamma + \gamma^o\right)^2\sigma_Y^2 = 0$ (which crosses 0 from below at $\gamma = \gamma_T^{\text{NX}}$, as stated in the proof of Lemma S.4).

Comparing (S.7) and (S.8) and using (S.10), it suffices to show that for all $t < T$,

$$\frac{2\sigma_Y^2}{2\sigma_Y^2 - \gamma_T^{\text{pub}}(T-t)} > \frac{\gamma_t^{\text{NX}}}{\gamma_T^{\text{NX}}} = \frac{\sigma_Y^2\left(\gamma^o + \gamma_T^{\text{NX}}\right)^2}{\sigma_Y^2\left(\gamma^o + \gamma_T^{\text{NX}}\right)^2 - (T-t)\left(\gamma^o\right)^2\gamma_T^{\text{NX}}}$$

$$\iff \frac{\sigma_Y^2}{\sigma_Y^2 - \frac{1}{2}\gamma_T^{\text{pub}}(T-t)} > \frac{\sigma_Y^2}{\sigma_Y^2 - \left(\frac{\gamma^o}{\gamma^o + \gamma_T^{\text{NX}}}\right)^2\gamma_T^{\text{NX}}(T-t)}.$$

In turn, it suffices to show that

$$\frac{1}{2}\gamma_T^{\text{pub}} > \left(\frac{\gamma^o}{\gamma^o + \gamma_T^{\text{NX}}}\right)^2\gamma_T^{\text{NX}},$$

which can be written as

$$\frac{\rho^{\text{pub}}}{2} > \frac{\rho^{\text{NX}}}{(1+\rho^{\text{NX}})^2}, \tag{S.11}$$

where $\rho^{\text{pub}} := \gamma_T^{\text{pub}}/\gamma^o \in (0,1)$ and $\rho^{\text{NX}} := \gamma_T^{\text{NX}}/\gamma^o \in (0,1)$. Note that $\rho^{\text{pub}} > \rho^{\text{NX}}$ by Lemma S.4. Moreover, dividing through the cubic $q(\gamma)$ by $(\gamma^o)^3\sigma_Y^2$, defining $\tilde{T} := \frac{T\gamma^o}{\sigma_Y^2}$, and changing variables, ρ^{NX} is the unique value of $\rho \in (0,1)$ solving the cubic

$$q(\tilde{T},\rho) := \tilde{T}\rho - (1-\rho)(1+\rho)^2 = 0,$$

where we make the dependence on $\tilde{T}$ explicit. Again, $q(\tilde{T},\cdot)$ crosses 0 from below at $\rho = \rho^{\text{NX}}$.

Observe that $\rho \mapsto \frac{\rho}{(1+\rho)^2}$ is strictly increasing on $(0,1)$. Define $\rho^* := \sqrt{2}-1 \in (0,1)$ and observe also that

- If $\rho \in [\rho^*, 1)$, then $\frac{\rho}{2} \geq \frac{\rho}{(1+\rho)^2}$.
- If $\rho \in (0, \rho^*)$, then $\frac{\rho}{2} < \frac{\rho}{(1+\rho)^2}$.

We prove (S.11) via two cases: (i) $\rho^{\text{pub}} \in [\rho^*, 1)$ and (ii) $\rho^{\text{pub}} \in (0, \rho^*)$.

Case (i): We have

$$\frac{\rho^{\text{pub}}}{2} \geq \frac{\rho^{\text{pub}}}{(1+\rho^{\text{pub}})^2} > \frac{\rho^{\text{NX}}}{(1+\rho^{\text{NX}})^2},$$

where the first inequality follows from $\rho^{\text{pub}} \geq \rho^*$ and the second from the fact that $0 < \rho^{\text{NX}} < \rho^{\text{pub}} < 1$, so we are done.

Case (ii): We have

$$0 < \frac{\rho^{\text{pub}}}{2} < \frac{\rho^{\text{pub}}}{(1+\rho^{\text{pub}})^2},$$

and thus there is a unique value $f(\rho^{\text{pub}}) = \frac{1-\rho^{\text{pub}}-\sqrt{1-2\rho^{\text{pub}}}}{\rho^{\text{pub}}}$ in the interval $(0, \rho^{\text{pub}})$ such that

$$\frac{\rho^{\text{pub}}}{2} = \frac{f(\rho^{\text{pub}})}{(1+f(\rho^{\text{pub}}))^2}.$$

We now show that $\rho^{\text{NX}} < f(\rho^{\text{pub}})$, which implies $\frac{\rho^{\text{pub}}}{2} = \frac{f(\rho^{\text{pub}})}{(1+f(\rho^{\text{pub}}))^2} > \frac{\rho^{\text{NX}}}{(1+\rho^{\text{NX}})^2}$, concluding the proof. Since $q(\tilde{T},\cdot)$ crosses 0 from below at ρ^{NX}, it suffices to show that

$$0 < q(\tilde{T}, f(\rho^{\text{pub}})). \tag{S.12}$$

for all $\tilde{T} > 0$ that induce $\rho^{\text{pub}} \in (0, \rho^*)$. Note that dividing (S.9) through by γ^o and simplifying the right hand side yields $\rho^{\text{pub}} = \rho^{\text{pub}}(\tilde{T}) = \frac{\tilde{T}+2-\sqrt{\tilde{T}^2+4}}{\tilde{T}}$, which has inverse $\tilde{T}(\rho^{\text{pub}}) = \frac{4(1-\rho^{\text{pub}})}{(2-\rho^{\text{pub}})\rho^{\text{pub}}}$, so we can establish (S.12) by equivalently showing that for all $\rho \in (0, \rho^*)$,

$$
\begin{aligned}
0 &< q(\tilde{T}(\rho), f(\rho)) \\
&= \frac{2}{(2-\rho)\rho^3}\left[5\rho^2 - 10\rho + 4 - (\rho^2 - 6\rho + 4)\sqrt{1-2\rho}\right].
\end{aligned}
$$

The outside factor is clearly positive, and by a change of variables $x = \sqrt{1-2\rho}$, so that $\rho = \frac{1-x^2}{2}$, the expression in square brackets simplifies to

$$g(x) = \frac{1}{4}(1-x)^5.$$

Note that each $\rho \in (0, \rho^*)$ is the image of some $x = \sqrt{1-2\rho}$ in $(\rho^*, 1)$, and over the latter domain, $g(x) > 0$, completing the proof.

S.1.6 Commitment Solution to Static Benchmark

Recall the static monetary policy game discussed in Section 2.

Proposition S.2. *The commitment solution to the static game is $a^c(\theta) = \frac{\theta+\mu}{2}$.*

Proof. Let p denote the density of $N(\mu, \gamma^o)$ (the distribution of θ), and let $R(\hat{a}, a, \theta) = -(k + \hat{a} - a)^2 - (a - \theta)^2$ denote the sender's ex post payoff function. The sender's problem is

$$\sup_{a(\cdot),\hat{a}} \int R(\hat{a}, a(\theta), \theta)p(\theta)\, d\theta$$

subject to the receiver playing a best response: $\hat{a} = \mathbb{E}[a(\theta)] = \int a(\theta)p(\theta)\, d\theta$.

The Lagrangian for the sender's problem is

$$\mathcal{L}(a(\cdot), \hat{a}, \lambda) = \int R(\hat{a}, a(\theta), \theta)p(\theta)\, d\theta + \lambda\left[\hat{a} - \int a(\theta)p(\theta)\, d\theta\right].$$

For each θ, the first order condition with respect to $a(\theta)$ is $R_{a(\theta)}p(\theta) - \lambda p(\theta) = 0$,

$$\implies \lambda = R_{a(\theta)} = -4a(\theta) + 2k + 2\hat{a} + 2\theta. \tag{S.13}$$

And the first order condition with respect to $\hat{a}$ is $\int R_{\hat{a}} p(\theta)\, d\theta + \lambda = 0$,

$$\implies \lambda = \int 2(\hat{a} + k - a(\theta))p(\theta)d\theta = 2(\hat{a} + k) - 2\int a(\theta)p(\theta)d\theta.$$

Using the constraint, this implies

$$\lambda = 2(\hat{a} + k) - 2\hat{a} = 2k.$$

Moreover, integrating (S.13) over all θ (with density p) and using the constraint yields

$$\lambda = -2\hat{a} + 2k + 2\mu,$$

and thus $\hat{a} = \mu$. Finally, plugging $\lambda = 2k$ and $\hat{a} = \mu$ into (S.13) gives the solution $a^c(\theta) = \frac{\theta+\mu}{2}$.

□

S.2 Reputation Game (Section 4.2): Omitted Proofs

Throughout this section, we again use NX to refer to the case $\sigma_X = +\infty$ where the X signal is pure noise, and we use pub to refer to the $\sigma_X = 0$ case.

S.2.1 Proof of Proposition 5

To sign coefficients, we work with ODEs in backward form. Consider any LME. To see that $\beta_0 \equiv 0$, just note that the terminal conditions imply $\beta_{0,0} = v_{1,0} = v_{3,0} = 0$, and moreover, $(\beta_0, v_1, v_3) = (0, 0, 0)$ satisfy the system of ODEs for these three variables. These are the unique solutions by the Picard-Lindelöf theorem.

Next, note that $(\gamma_0, \chi_0) \in (0, \gamma^o) \times (0, 1)$. As in the proof of Theorem 1, define $(\tilde{\beta}_2, \tilde{v}_6, \tilde{v}_8) := (\beta_2/(1-\chi), v_6\gamma/(1-\chi)^2, v_8\gamma/(1-\chi))$; also, define $\tilde{\beta}_3 := \beta_3 - 1$. Using the initial values $\beta_{1,0} = -\frac{\psi\gamma_0}{\sigma_Y^2 + \psi\gamma_0\chi_0} < 0$, and $\tilde{\beta}_{2,0} = \tilde{\beta}_{3,0} = \tilde{v}_{60} = \tilde{v}_{80} = 0$, it is tedious but straightforward to verify that $\dot{\tilde{\beta}}_{2,0} < 0$, $\dot{\tilde{\beta}}_{3,0} < 0$, $\dot{\tilde{v}}_{60} = 0 > \ddot{\tilde{v}}_{60}$, and $\dot{\tilde{v}}_{80} = 0 > \ddot{\tilde{v}}_{80}$. (See `spm.nb` on our websites.) Hence, for all sufficiently small $t > 0$, for all $x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}$, we have $x_t < 0$. Define $\tau := \inf\{t \in (0, T] : x_t = 0 \text{ for some } x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}\}$ (and $\tau = \infty$ if this set is empty). Suppose by way of contradiction that $\tau \leq T$. By continuity, $x_\tau = 0$ for some x. We derive a contradiction by arguing via the comparison theorem that for all $t \in (0, \tau]$ and all $x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}$, $x_t < 0$. Write each ODE in the form $\dot{x}_t = f^x(x_t, t)$, and define $y \equiv 0$. Consider any $s \in (0, \tau)$; by the definition of τ, each variable is strictly negative over $(0, s]$, so in particular, for each x, we have $x_s < y_s = 0$. And by definition, $\dot{x}_t - f^x(x_t, t) = 0$ over

$[0, T]$. Moreover, for $t \in [s, \tau]$,

$$
\begin{aligned}
\dot{y}_t - f^{\beta_1}(y_t, t) &= -\frac{2\tilde{\beta}_2 \gamma_t \chi_t}{\sigma_X^2} \geq 0 \\
\dot{y}_t - f^{\tilde{\beta}_2}(y_t, t) &= \frac{\alpha_{3t}\gamma_t(\sigma_X^2 \beta_{1t}^2 - 2\tilde{v}_{6t}\chi_t)}{\sigma_X^2 \sigma_Y^2} \geq 0 \\
\dot{y}_t - f^{\tilde{\beta}_3}(y_t, t) &= -\frac{\gamma_t(1 + \beta_{1t}\chi_t)[\sigma_X^2 \beta_{1t} + \tilde{v}_{8t}\chi_t]}{\sigma_X^2 \sigma_Y^2} \geq 0 \\
\dot{y}_t - f^{\tilde{v}_6}(y_t, t) &= \frac{1}{2}\tilde{\beta}_{2t}\gamma_t(2\beta_{1t} + \tilde{\beta}_{2t}) \geq 0 \\
\dot{y}_t - f^{\tilde{v}_8}(y_t, t) &= -\gamma_t[\tilde{\beta}_{2t} - \beta_{1t}\tilde{\beta}_{3t}] \geq 0,
\end{aligned}
$$

where we have used that $\alpha_3 > 0$ (since $\alpha_{3,0} = \frac{\sigma_Y^2}{\sigma_Y^2 + \psi\gamma_0\chi_0} > 0$ and α_3 does not change sign, as shown in the proof of Theorem 1), that for all $x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}$ and all $t \in [0, \tau]$, $x_t \leq 0$, and in the third line, that $1 + \beta_{1t}\chi_t \geq \beta_{3t} + \beta_{1t}\chi_t = \alpha_{3t} > 0$. By the comparison theorem, we have $x_t < y_t = 0$ for all $t \in [s, \tau]$ and all $x \in \{\beta_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{v}_6, \tilde{v}_8\}$, contradicting that $x_\tau = 0$ for some such x. Hence, $\tau = \infty$, and we conclude that for all $t > 0$ going backward ($t < T$ going forward), $\beta_{1t} < 0$, $\beta_{2t} = \tilde{\beta}_{2t}(1 - \chi_t) < 0$, and $\beta_{3t} = \tilde{\beta}_{3t} + 1 < 1$, from which it follows that $\beta_{3t} = \alpha_{3t} - \beta_{1t}\chi_t \geq \alpha_{3t} > 0$. Moreover, $\alpha_{3t} = \beta_{3t} + \beta_{1t}\chi_t \leq \beta_{3t} < 1$ for $t \in [0, T)$. The remaining inequalities at time T (going forward) are immediate from the terminal conditions.

S.2.2 Proof of Proposition 6

Assume throughout that $r = 0$ and $\psi < \sigma_Y^2/\gamma^o$. We first characterize the unique equilibria for the cases $\sigma_X = 0$ and $\sigma_X = +\infty$, and then we compare the sender's payoffs.

Public Case $\sigma_X = 0$

We look for an equilibrium of the form $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{3t}\theta$, where $M_t = \hat{M}_t$ is publicly known, with value function $V(\theta, m, t) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\theta^2 + v_{4t}m^2 + v_{5t}\theta m$.

The core (backward) system of ODEs is

$$
(\dot{\beta}_{0t}, \dot{\beta}_{1t}, \dot{\beta}_{3t}, \dot{\gamma}_t) = (0, -\beta_{1t}\beta_{3t}^2\gamma_t^2/\sigma_Y^2, \beta_{1t}\beta_{3t}^2\gamma_t^2/\sigma_Y^2, \beta_{3t}^2\gamma_t^2/\sigma_Y^2).
$$

with initial conditions $\beta_{0,0} = 0, \beta_{1,0} = -\frac{\psi\gamma_0}{\sigma_Y^2} \leq 0, \beta_{3,0} = 1$ and $\gamma_0 = \gamma^F \in (0, \gamma^o)$.

Define $\tilde{\psi} := \psi\gamma^o/\sigma_Y^2 < 1$ and $\tilde{T} := T\gamma^o/\sigma_Y^2$.

We now show that there is a unique $\gamma^F \in (0, \gamma^o)$ such that a (unique) solution to the IVP exists and satisfies $\gamma_T = \gamma^o$, and thus there exists a unique LME. Specifically, $\gamma^F = \rho^{\text{pub}}\gamma^o$, where ρ^{pub} is the unique root of the cubic $g^{\text{pub}}(\rho) := -\tilde{T}\tilde{\psi}\rho^2(1-\rho) + \rho(1+\tilde{T}) - 1 = 0$; and $(\beta_{0t}, \beta_{1t}, \beta_{3t}) = (0, \beta_{1,0}\gamma^F/\gamma_t, 1 + \beta_{1,0}(1-\gamma^F/\gamma_t))$ where $\gamma_t = \frac{\gamma^F[\sigma_Y^4 + t\psi(\gamma^F)^2]}{\sigma_Y^4 - t\gamma^F(-\gamma^F\psi + \sigma_Y^2)}$. (It is easy to verify that $g^{\text{pub}}(0) < 0 < g^{\text{pub}}(1)$, and moreover, $\tilde{\psi} < 1$ implies $g^{\text{pub}\prime}(\rho) > 0$ for all $\rho \in \mathbb{R}$, so there is indeed a unique root of the cubic.)

Note that that $\beta_0 = 0$ is the unique solution to its ODE and initial condition. Now $\dot{\beta}_{1t} + \dot{\beta}_{3t} = 0$, so $\beta_1 + \beta_3$ is constant, and

$$\beta_{1t} + \beta_{3t} = \beta_{10} + \beta_{30} = 1 + \beta_{10} \implies \beta_{1t} = 1 + \beta_{10} - \beta_{3t}.$$

Next, define $\Pi := \beta_1\gamma$ and observe that $\dot{\Pi} \equiv 0$, so

$$\begin{aligned} &\beta_{1t}\gamma_t = \beta_{10}\gamma^F \\ \implies\ &\beta_{1t} = \beta_{10}\gamma^F/\gamma_t \\ \implies\ &\beta_{3t} = 1 + \beta_{10}(1 - \gamma^F/\gamma_t), \end{aligned} \tag{S.14}$$

where $\gamma_t \geq \gamma^F > 0$ for all t over the interval of existence, since γ is nondecreasing.

Using (S.14), the ODE for γ is $\dot{\gamma}_t = [(1+\beta_{10})\gamma_t - \beta_{10}\gamma^F]^2/\sigma_Y^2$. Integrating and using the initial condition for β_{10} and $\gamma_0 = \gamma^F$ yields

$$\gamma_t = \frac{\gamma^F[\sigma_Y^4 + t\psi(\gamma^F)^2]}{\sigma_Y^4 - t\gamma^F(-\gamma^F\psi + \sigma_Y^2)},$$

wherever this exists. The condition $\gamma_T = \gamma^o$, after writing $\gamma^F = \rho\gamma^o$, is equivalent to the cubic equation $g^{\text{pub}}(\rho) = 0$ stated above, completing the characterization of LME.

$\sigma_X = +\infty$ Case

We look for an equilibrium with $a_t = \beta_{0t}\mu + \beta_{1t}M_t + \beta_{3t}\theta$, where $M_t = \mathbb{E}_t[\hat{M}_t]$, with value function $V(t,\theta,m) = v_{0t} + v_{1t}\theta + v_{2t}m + v_{3t}\theta^2 + v_{4t}m^2 + v_{5t}\theta m$. The backward system is

$$\begin{aligned} \dot{\beta}_0 &= -\frac{\beta_1^2\gamma(1-\chi)\alpha_3}{\sigma_Y^2} \\ \dot{\beta}_1 &= -\frac{\beta_1\gamma(\alpha_3 - \beta_1)\alpha_3}{\sigma_Y^2} \\ \dot{\beta}_3 &= \frac{\beta_3\beta_1\gamma\alpha_3}{\sigma_Y^2} \\ \dot{\gamma}_t &= \frac{\gamma^2\alpha_3^2}{\sigma_Y^2} \end{aligned}$$

where $\chi = \chi(\gamma) = 1 - \gamma/\gamma^o$, and with initial conditions

$$\beta_{00} = 0, \beta_{10} = -\frac{\psi\gamma^F}{\sigma_Y^2 + \psi\gamma^F\chi(\gamma^F)} < 0, \beta_{30} = 1, \text{ and } \gamma_0 = \gamma^F \in (0, \gamma^o).$$

Also note that $\dot{\alpha}_{3t} = 0$, so $\alpha_3 = \alpha_{3,0} = \bar{\alpha} := \frac{\sigma_Y^2}{\sigma_Y^2 + \psi\gamma_0\chi(\gamma_0)}$, and thus it suffices to solve the γ-ODE with terminal condition $\gamma_T = \gamma^o$ to pin down γ^F. (The remaining value function coefficients are easily recovered as in the previous application.)

We show that there is one solution for each root $\rho^{\text{NX}} \in (0,1)$ of the quintic $g^{\text{NX}}(\rho) := \tilde{T}\rho - (1-\rho)[1 + \tilde{\psi}\rho(1-\rho)]^2$, where $\tilde{T} := T\gamma^o/\sigma_Y^2$ and $\tilde{\psi} := \psi\gamma^o/\sigma_Y^2$. Such a root always exists since $g^{\text{NX}}(0) < 0 < g^{\text{NX}}(1)$, and hence an LME exists. In any LME, $\gamma_t = \frac{\gamma^F\sigma_Y^2}{\sigma_Y^2 - \gamma^F\alpha_3^2 t}$, where $\gamma^F = \rho^{\text{NX}}\gamma^o$, and with $\alpha_3 = \alpha_{3,0}$ being a constant as above. We then show that there is a unique root in $(0,1)$ when $\tilde{\psi} < 1$.

To establish these claims, consider the backward IVP indexed by γ^F over its maximal interval of existence. Notice first that $\dot{\beta}_0 + \dot{\beta}_1 + \dot{\beta}_3 \equiv 0$, and so

$$\beta_{0t} + \beta_{1t} + \beta_{3t} = \beta_{00} + \beta_{10} + \beta_{30} = 1 - \frac{\psi\gamma^F}{\sigma_Y^2 + \psi\gamma^F\chi(\gamma^F)}.$$

Thus, as long as β_1 and β_3 exist, β_0 will too, and since β_0 does not appear in any of the other ODEs, we can ignore it from the analysis.

Now consider the subsystem

$$\dot{\beta}_1 = -\frac{\beta_1\bar{\alpha}\gamma(\bar{\alpha} - \beta_1)}{\sigma_Y^2}$$
$$\dot{\beta}_3 = \frac{\beta_3\beta_1\gamma\bar{\alpha}}{\sigma_Y^2},$$

and observe that since $\beta_{10} < 0$ and $\beta_{30} = 1 > 0$, the same inequalities hold in a neighborhood of zero.

We claim that β_3 and β_1 do not change signs. First, both cannot vanish at the same time, as this would violate that $\alpha_{3t} = \bar{\alpha} > 0$. Now suppose β_3 is the first to do it, say at time t; then for all $s \in [0,t]$, $\beta_{10} < 0$ and by the comparison theorem, $\beta_3 > 0$ for all $s \in [0,t]$, a contradiction. Likewise, a contradiction obtains if β_1 vanishes first. We therefore conclude that β_1 is increasing while β_3 is decreasing, and that they lie in $[\beta_{10}, 0]$ and $[0,1]$ as long as they exist.

The existence of a solution to the IVP over $[0,T]$ then reduces to the existence of a solution to the γ-ODE when this ODE is driven by $\bar{\alpha}$. As long as it exists, straightforward integration shows that

$$\gamma_t = \frac{\gamma^F \sigma_Y^2}{\sigma_Y^2 - \gamma^F \bar{\alpha}^2 t}.$$

Imposing $\gamma_T = \gamma^o$ and using that $\bar{\alpha} = \sigma_Y^2/[\sigma_Y^2 + \psi\gamma^F(1-\gamma^F/\gamma^o)] = 1/[1+\tilde{\psi}\rho(1-\rho)]$ yields the quintic equation $g^{NF}(\rho) = 0$ introduced earlier. (Note that when $\gamma_T = \gamma^o$, γ_t is well defined for all $t \in [0,T]$.)

To show uniqueness for $\tilde{\psi} < 1$, we prove that the derivative of g^{NF} is positive at any point in $(0,1)$ that satisfies $g^{NF}(\rho) = 0$; thus, g^{NF} can only cross zero once, and hence, it does so from below. It is easy to verify that

$$(g^{NF})'(\rho) = \tilde{T} + [1+\tilde{\psi}\rho(1-\rho)]^2 - 2\tilde{\psi}(1-\rho)(1-2\rho)[1+\tilde{\psi}\rho(1-\rho)].$$

At a crossing point, however, $\tilde{T} + [1+\tilde{\psi}\rho(1-\rho)]^2 = \frac{[1+\tilde{\psi}\rho(1-\rho)]^2}{\rho}$, and so

$$\begin{aligned}(g^{NF})'(\rho) &= \frac{1}{\rho}[1+\tilde{\psi}\rho(1-\rho)]\left\{1+\tilde{\psi}\rho(1-\rho) - 2\tilde{\psi}\rho(1-\rho)(1-2\rho)\right\}\\ &\geq \frac{1}{\rho}[1+\tilde{\psi}\rho(1-\rho)]\left\{1+0-2\tilde{\psi}\frac{1}{4}\right\} > 0,\end{aligned}$$

where we have used that $0 < \tilde{\psi} < 1$, $0 < \rho(1-\rho) \leq 1/4$, and $|1-2\rho| \leq 1$.

Payoff Comparisons

The following lemma will be useful for comparing the sender's payoffs.

Lemma S.5. *If $\tilde{\psi} \in (0,1)$, then there is more learning in the public case for all $T > 0$.*

Proof. Let $\rho^x = \gamma_T^x/\gamma^o \in (0,1)$, where γ_T^x is the terminal value of γ in the BVP of case $x \in \{\text{pub}, \text{NX}\}$. When $\tilde{\psi} \in (0,1)$, these values are the unique roots of

$$\begin{aligned}0 &= g^{\text{NX}}(\rho) := \rho\tilde{T} - (1-\rho)[1+\tilde{\psi}\rho(1-\rho)]^2 = \rho(1+\tilde{T}) - 1 - \tilde{\psi}\rho(1-\rho)^2[2+\tilde{\psi}\rho(1-\rho)]\\ 0 &= g^{\text{pub}}(\rho) := \rho(1+\tilde{T}) - 1 - \tilde{\psi}\tilde{T}\rho^2(1-\rho),\end{aligned}$$

respectively. In particular, observe that $\rho^x > 1/(1+\tilde{T})$, $x \in \{\text{pub}, \text{NX}\}$. Our goal is to show $\rho^{\text{pub}} < \rho^{\text{NX}}$.

Now, using that $\rho^{\text{pub}}(1+\tilde{T}) - 1 = \tilde{\psi}\tilde{T}(\rho^{\text{pub}})^2(1-\rho^{\text{pub}})$, we get that

$$g^{\text{NX}}(\rho^{\text{pub}}) = \frac{\tilde{\psi}(1-\rho^{\text{pub}})}{\tilde{T}}\left\{\tilde{T}^2(\rho^{\text{pub}})^2 - (1-\rho^{\text{pub}})[2\rho^{\text{pub}}\tilde{T} + \rho^{\text{pub}}(1+\tilde{T}) - 1]\right\}.$$

where $\frac{\tilde{\psi}(1-\rho^{\text{pub}})}{\tilde{T}} > 0$. Thus, letting

$$Q(\rho) := \tilde{T}^2\rho^2 - (1-\rho)[2\rho\tilde{T} + \rho(1+\tilde{T}) - 1] = \rho^2(\tilde{T}^2 + 3\tilde{T} + 1) - \rho(3\tilde{T} + 2) + 1,$$

it suffices to show that $Q(\rho^{\text{pub}}) < 0$, as $g^{\text{NX}}(\rho) < 0$ if and only if $\rho < \rho^{\text{NX}}$.

Observe that the roots of Q are given by $\rho_- := \frac{(3-\sqrt{5})\tilde{T}+2}{2(\tilde{T}^2+3\tilde{T}+1)}$ and $\rho_+ := \frac{(3+\sqrt{5})\tilde{T}+2}{2(\tilde{T}^2+3\tilde{T}+1)}$, and that $\rho_- < \frac{1}{1+\tilde{T}} < \rho_+$. Consequently, it suffices to show that $g^{\text{pub}}(\rho_+) > 0$: this ensures that $\rho^{\text{pub}} < \rho_+$, and since $\rho^{\text{pub}} > \frac{1}{1+\tilde{T}} > \rho_-$, this implies that $Q(\rho^{\text{pub}}) < 0$.

Straightforward algebraic manipulation yields that $g^{\text{pub}}(\rho_+) > 0$ if and only if

$$\begin{aligned}\tilde{g}(\tilde{T},\tilde{\psi}) \;&:= 4(1+\tilde{T})[(3+\sqrt{5})\tilde{T}+2][\tilde{T}^2+3\tilde{T}+1]^2 - 8[\tilde{T}^2+3\tilde{T}+1]^3 \\ &\quad -\tilde{\psi}\tilde{T}^2[(3+\sqrt{5})\tilde{T}+2]^2[2\tilde{T}+(3-\sqrt{5})] > 0.\end{aligned}$$

A lower bound on the left hand side is found by setting $\psi = 1$, and $\tilde{g}(\tilde{T}, 1)$ can be written as $\tilde{T}\sum_{i=0}^{5} a_i\tilde{T}^i$ where all the $a_i > 0$. Hence, $\tilde{g}(\tilde{T},\tilde{\psi}) > 0$ whenever $\tilde{T} > 0$ and $\tilde{\psi} \in (0, 1]$, concluding the proof. □

We now leverage Lemma S.5 to compare ex ante payoffs. To simplify expressions, we again rescale payoffs to remove the outside scalar factor of $\frac{1}{2}$. Let V^x denote the ex ante payoff to the politician in the case $x \in \{\text{pub}, \text{NX}\}$. First,

$$\begin{aligned}V^{\text{pub}} &= \mathbb{E}_0\left[-\int_0^T (a_t-\theta)^2 dt - \psi M_T^2\right] \\ &= -\int_0^T \mathbb{E}_0\left[(\beta_{1t}M_t + [\beta_{3t}-1]\theta)^2\right] dt - \psi(\mu^2 + \gamma^o - \gamma_T) \\ &= -\int_0^T \left[(\beta_{3t}-1)^2\gamma^o + \beta_{1t}^2(\gamma^o-\gamma_t) + 2\beta_{1t}(\beta_{3t}-1)(\gamma^o-\gamma_t)\right] dt - \tilde{\psi}\sigma_Y^2(1-\rho^{\text{pub}}).\end{aligned}$$

Using the solutions for the coefficients and γ_t in terms of γ^F and carrying out the simplifications, we obtain $V^{\text{pub}} = V^{\text{pub}}(\rho^{\text{pub}})$, where

$$V^{\text{pub}}(\rho) := \sigma_Y^2\left\{-\tilde{\psi}(1-\rho) + \tilde{T}\tilde{\psi}\rho^2[-\tilde{\psi}(1-\rho)+1] + \ln\left(\frac{1-\rho}{\tilde{T}\rho}\right)\right\}.$$

In the $\sigma_X = +\infty$ case, note that $\mathbb{E}_0[M_t^2] = \mathbb{E}_0[(\chi_t\theta + (1-\chi_t)\mu)^2] = \mathbb{E}_0[\chi_t^2\theta^2] = \chi_t^2\gamma^o$. Hence, $\mathbb{E}_0[\hat{M}_t^2] = \mathbb{E}_0[(\hat{M}_t - M_t)^2] + \mathbb{E}_0[M_t^2] = \gamma_{2t} + \chi_t^2\gamma^o = \chi_t\gamma_t + \chi_t^2\gamma^o$.

Using $a_t = \beta_{0t} + \beta_{1t} M_t + \beta_{3t}\theta = \bar{a}\theta$, we now calculate

$$V^{\mathrm{NX}} = \mathbb{E}_0 \left[-\int_0^T (a_t - \theta)^2 dt - \psi(\chi_T \gamma_T + \chi_T^2 \gamma^o) \right] = -(1-\alpha_3)^2 \gamma^o T - \psi \chi_T (\gamma_T + \chi_T \gamma^o).$$

Expressing $\chi_T = 1 - \gamma_T/\gamma^o$, γ_T and α_3 in terms of $\gamma^F = \gamma_T$, we have $V^{\mathrm{NX}} = V^{\mathrm{NX}}(\rho^{\mathrm{NX}})$, where

$$V^{\mathrm{NX}}(\rho) := \sigma_Y^2 \left\{ -\tilde{\psi}^2 \rho (1-\rho)^3 - \tilde{\psi}(1-\rho) \right\}.$$

We now prove that for $r = 0$ and $\tilde{\psi} \in (0,1)$, the sender's ex ante payoff is higher for $\sigma_X = +\infty$ than for $\sigma_X = 0$. To do so, we show that (i) $V^{\mathrm{pub}}(\rho^{\mathrm{pub}}) < V^{\mathrm{NX}}(\rho^{\mathrm{pub}})$ and (ii) $V^{\mathrm{NX}}(\rho)$ is increasing for $\rho \geq \rho^{\mathrm{pub}}$. Since $\rho^{\mathrm{pub}} < \rho^{\mathrm{NX}}$ by Lemma S.5, it follows that $V^{\mathrm{pub}}(\rho^{\mathrm{pub}}) < V^{\mathrm{NX}}(\rho^{\mathrm{NX}})$.

Toward establishing (i), define $\tilde{V}(\rho) := V^{\mathrm{pub}}(\rho) - V^{\mathrm{NX}}(\rho)$; we have

$$\tilde{V}(\rho) = \sigma_Y^2 \left\{ \tilde{T}\tilde{\psi}\rho^2[-\tilde{\psi}(1-\rho)+1] + \ln\left(\frac{1-\rho}{\tilde{T}\rho}\right) + \tilde{\psi}^2 \rho (1-\rho)^3 \right\},$$

and our first goal is to show $\tilde{V}(\rho^{\mathrm{pub}}) < 0$. Since $\ln(x) < x - 1$ for $x > 0$, we have

$$\tilde{V}(\rho) < \sigma_Y^2 \left\{ \tilde{T}\tilde{\psi}\rho^2[-\tilde{\psi}(1-\rho)+1] + \left[\frac{1-\rho}{\tilde{T}\rho} - 1\right] + \tilde{\psi}^2 \rho (1-\rho)^3 \right\} = \frac{\sigma_Y^2}{\tilde{T}\rho} \tilde{V}_2(\rho),$$

where $\tilde{V}_2(\rho) := \tilde{T}^2 \tilde{\psi} \rho^3 [1 - \tilde{\psi}(1-\rho)] + 1 - \rho(1+\tilde{T}) + \tilde{T}\tilde{\psi}^2 \rho^2 (1-\rho)^3$, and so it suffices to show $\tilde{V}_2(\rho^{\mathrm{pub}}) < 0$. Now the equation $g^{\mathrm{pub}}(\rho^{\mathrm{pub}}) = 0$ is equivalent to $\tilde{\psi} = -\frac{1-(1+\tilde{T})\rho}{\tilde{T}\rho^2(1-\rho)}|_{\rho=\rho^{\mathrm{pub}}}$; using this to eliminate $\tilde{\psi}$ and simplifying, we obtain $\tilde{V}_2(\rho^{\mathrm{pub}}) = -\frac{[\rho(1+\tilde{T})-1]^3}{\tilde{T}\rho^2}|_{\rho=\rho^{\mathrm{pub}}}$, which is strictly negative as $\rho^{\mathrm{pub}} > \frac{1}{1+\tilde{T}}$, establishing claim (i).

Toward claim (ii), differentiate

$$\frac{d}{d\rho} V^{\mathrm{NX}}(\rho) = \sigma_Y^2 \left\{ -\tilde{\psi}^2 [-3\rho(1-\rho)^2 + (1-\rho)^3] + \tilde{\psi} \right\} = \sigma_Y^2 \tilde{\psi} \left\{ -\tilde{\psi}(1-\rho)^2(1-4\rho) + 1 \right\}.$$

The expression in braces is positive iff $h(\rho) := (1-\rho)^2(1-4\rho) < \frac{1}{\tilde{\psi}}$. Now for $\rho \in [0,1]$, $h(\rho)$ attains its maximum value of 1 at $\rho = 0$. Hence, if $\tilde{\psi} \leq 1$, the expression is positive for all $\rho \in (0,1)$ and we conclude that $V^{\mathrm{NX}}(\rho)$ is increasing for all $\rho \geq \rho^{\mathrm{pub}}$.

Combining parts (i) and (ii) yields $V^{\mathrm{pub}}(\rho^{\mathrm{pub}}) < V^{\mathrm{NX}}(\rho^{\mathrm{pub}}) < V^{\mathrm{NX}}(\rho^{\mathrm{NX}})$ as desired.

S.3 Existence of Linear Markov Equilibria (Section 5): Omitted Proofs

S.3.1 Auxiliary Results

In this section we prove Lemma C.3 from the main text.

Proof of Lemma C.3. Let $(x_t^i)_{t\in[0,T]}$ denote the solution to $IVP(y^i)$ from the statement of Lemma C.3. For each component $j \in \{1,\dots,n\}$, the triangle inequality implies

$$|x_{jt}^1 - x_{jt}^2| \le |\omega_j(y_0^1) - \omega_j(y_0^2)| + \int_0^t |F_j(x_s^1, y_s^1) - F_j(x_s^2, y_s^2)| ds.$$

The mean value theorem and the facts that F_j is of class C^1 and X and Y are compact then yield that there exists $c^j \in \mathbb{R}_+$ such that for all $y^1, y^2 \in \mathcal{Y}$,

$$|x_{jt}^1 - x_{jt}^2| \le |\omega_j(y_0^1) - \omega_j(y_0^2)| + c_j \int_0^t ||y_s^1 - y_s^2||_\infty ds + c_j \int_0^t ||x_s^1 - x_s^2||_\infty ds.$$

Letting $c = \max\{c_j : j = 1, ..., n\}$, we obtain

$$||x_t^1 - x_t^2||_\infty \le ||\omega(y_0^1) - \omega(y_0^2)||_\infty + c\int_0^t ||y_s^1 - y_s^2||_\infty ds + c\int_0^t ||x_s^1 - x_s^2||_\infty ds.$$

Since $c \ge 0$ and $t \mapsto ||\omega(y_0^1) - \omega(y_0^2)||_\infty + c\int_0^t ||y_s^1 - y_s^2||_\infty ds$ is non-decreasing, Gronwall's inequality (Teschl, 2012, Lemma 2.7) implies that

$$\begin{aligned}
||x_t^1 - x_t^2||_\infty &\le e^{ct}\left(||\omega(y_0^1) - \omega(y_0^2)||_\infty + c\int_0^t ||y_s^1 - y_s^2||_\infty ds\right) \\
&\le e^{cT}\left((||\omega(y_0^1) - \omega(y_0^2)||_\infty + cT \sup_{s\in[0,T]} ||y_s^1 - y_s^2||_\infty\right) \\
&= k_1||\omega(y_0^1) - \omega(y_0^2)||_\infty + k_2 \sup_{s\in[0,T]} ||y_s^1 - y_s^2||_\infty.
\end{aligned}$$

□

S.3.2 Terminal Conditions ($\psi \neq 0$)

In this section, we provide a lemma characterizing terminal conditions, which we will use to prove Corollary C.2 in the next section.

Suppose the sender receives a terminal payoff of the form $\psi(\hat{a}_T) = \psi_1\hat{a}_T + \frac{1}{2}\psi_2\hat{a}_T^2$, where ψ_1, ψ_2 are constants. (An intercept is strategically irrelevant.) The following result gives sufficient conditions for the terminal game parameterized by (γ_T, χ_T) to have a unique equilibrium, continuously differentiable in those parameters, and it provides expressions for the terminal conditions for strategy coefficients (up to α_T, which is defined implicitly). The bound C_ψ/γ^o on the curvature, characterized in the proof of the lemma, ensures uniqueness by limiting strategic complementarities. Note that there are no restrictions on ψ_1.

Lemma S.6. *Under Assumptions 1 and 2, there exists $C_\psi \in (-\infty, 0)\cup\{-\infty\}$ independent of (r, γ^o) such that for all $(\psi_1, \psi_2) \in \mathbb{R} \times (C_\psi/\gamma^o, 0]$ there exists $\vec{\beta}_T(\gamma_T, \chi_T)$ continuously differentiable over $[0, \gamma^o] \times [0, 1]$ that, together with the receiver's myopic best reply, characterizes the unique Bayes Nash equilibrium of the terminal game parameterized by (γ_T, χ_T):*

$$\beta_{0T} = \frac{\sigma_Y^2[(u_{a0} + u_{a\hat{a}}\hat{u}_{\hat{a}0}) + (\psi_1 + \hat{u}_{\hat{a}0}\psi_2)(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})\alpha_{3T}\gamma_T/\sigma_Y^2]}{\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}) - \psi_2\gamma_T\alpha_{3T}\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})} \tag{S.15}$$

$$\beta_{1T} = (\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})\left(u_{a\hat{a}} + \psi_2(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})\alpha_{3T}\gamma_T/\sigma_Y^2\right) \tag{S.16}$$

$$\beta_{2T} = \frac{\sigma_Y^2\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})(1 - \chi_T)\left(u_{a\hat{a}} + \psi_2\alpha_{3T}\gamma_T(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})/\sigma_Y^2\right)^2}{\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}) - \psi_2\gamma_T\alpha_{3T}\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})} \tag{S.17}$$

$$\beta_{3T} = u_{a\theta}. \tag{S.18}$$

Moreover, $\alpha_{3T}(\gamma_T, \chi_T) := \beta_{1T}(\gamma_T, \chi_T)\chi_T + \beta_{3T}(\gamma_T, \chi_T)$ has the same sign as $u_{a\theta}$, and therefore the same sign as α_3^m. The value of C_ψ depends on the parameter values as follows:

- *If $\hat{u}_{a\hat{a}}\hat{u}_{a\theta} = 0$, then $C_\psi = -\infty$.*
- *If $\hat{u}_{a\hat{a}}\hat{u}_{a\theta} \neq 0$, then $C_\psi = -\frac{3\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\hat{u}_{\hat{a}\theta}^2} < 0$.*

Proof. We first derive the system of equations that characterize any Bayes Nash equilibrium of the static game at time T. Given that (i) the receiver plays $\hat{a}_t = \delta_{0t} + \delta_{1t}\hat{M}_t + \delta_{2t}L_t$, where $\delta_{0t} = \hat{u}_0 + \hat{u}_{a\hat{a}}\beta_{0t}$, $\delta_{1t} = \hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}(\beta_{3t} + \beta_{1t}\chi_t)$ and $\delta_{2t} = \hat{u}_{a\hat{a}}[\beta_{2t} + \beta_{1t}(1 - \chi_t)]$, and (ii) $M_t = \mathbb{E}_t[\hat{M}_t]$, all $t \in [0, T]$, imposing that the sender's strategy $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{2t}L_t + \beta_{3t}\theta$ satisfies the first-order condition on the right hand side of the HJB equation for times $t \in [0, T)$, we obtain the following equations:

$$\gamma_t\alpha_{3t}v_{2t} = -\sigma_Y^2[u_{a0} + u_{a\hat{a}}\hat{u}_{\hat{a}0} - (1 - u_{a\hat{a}}\hat{u}_{a\hat{a}})\beta_{0t}] \tag{S.19}$$

$$\gamma_t\alpha_{3t}v_{5t} = -\frac{\sigma_Y^2}{2}[u_{a\hat{a}}\hat{u}_{\hat{a}\theta} + u_{a\hat{a}}\hat{u}_{a\hat{a}}\alpha_{3t} - \beta_{1t}] \tag{S.20}$$

$$\gamma_t\alpha_{3t}v_{7t} = -\sigma_Y^2[u_{a\theta} - \beta_{3t}] \tag{S.21}$$

$$\gamma_t\alpha_{3t}v_{9t} = -\sigma_Y^2[u_{a\hat{a}}\hat{u}_{a\hat{a}}\beta_{1t}(1 - \chi_t) - \beta_{2t}(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}})]. \tag{S.22}$$

By continuity (wrt time) of the strategy, learning, and value function coefficients, (S.19)-(S.22) also hold at time $t = T$.

The sender's time-T expectation of the terminal payoff is

$$\mathbb{E}_T[\psi(\hat{a}_T)] = \psi_1[\delta_{0T} + \delta_{1T}M_T + \delta_{2T}L_T] + \frac{\psi_2}{2}[\delta_{0T} + \delta_{1T}M_T + \delta_{2T}L_T]^2 + \frac{\psi_2}{2}\delta_{1T}^2\gamma_T\chi_T,$$

from which we obtain

$$v_{2T} = [\psi_1 + \psi_2(\hat{u}_{\hat{a}0} + \hat{u}_{a\hat{a}}\beta_{0T})](\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T}) \tag{S.23}$$

$$v_{5T} = \frac{\psi_2}{2}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})^2 \tag{S.24}$$

$$v_{7T} = 0 \tag{S.25}$$

$$v_{9T} = \psi_2\hat{u}_{a\hat{a}}[\beta_{2T} + \beta_{1T}(1 - \chi_T)](\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T}). \tag{S.26}$$

For later use in our boundary value problem, we also note the terminal conditions

$$v_{6T} = \frac{\psi_2}{2}\delta_{2T}^2 = \frac{\psi_2}{2}\hat{u}_{a\hat{a}}^2[\beta_{2T} + \beta_{1T}(1 - \chi_T)]^2$$
$$v_{8T} = 0$$

Evaluating (S.19)-(S.22) at time $t = T$ and equating these with $(\gamma_T\alpha_{3T})$ times (S.23)-(S.26), respectively, we obtain

$$-\sigma_Y^2[u_{a0} + u_{a\hat{a}}\hat{u}_{\hat{a}0} - (1 - u_{a\hat{a}}\hat{u}_{a\hat{a}})\beta_{0T}] = \gamma_T\alpha_{3T}[\psi_1 + \psi_2(\hat{u}_{\hat{a}0} + \hat{u}_{a\hat{a}}\beta_{0T})](\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T}) \tag{S.27}$$

$$-\frac{\sigma_Y^2}{2}[u_{a\hat{a}}\hat{u}_{\hat{a}\theta} + u_{a\hat{a}}\hat{u}_{a\hat{a}}\alpha_{3T} - \beta_{1T}] = \frac{\psi_2}{2}\gamma_T\alpha_{3T}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})^2 \tag{S.28}$$

$$-\sigma_Y^2[u_{a\theta} - \beta_{3T}] = 0 \tag{S.29}$$

$$-\sigma_Y^2[u_{a\hat{a}}\hat{u}_{a\hat{a}}\beta_{1T}(1 - \chi_T) - \beta_{2T}(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}})] = \psi_2\gamma_T\alpha_{3T}\hat{u}_{a\hat{a}}[\beta_{2T} + \beta_{1T}(1 - \chi_T)](\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T}). \tag{S.30}$$

First, we characterize α_{3T}, and then we show that $(\beta_{0T}, \beta_{1T}, \beta_{2T}, \beta_{3T})$ are given by (S.15)-(S.18).

Multiplying (S.28) through by $2\chi_T$, substituting $\beta_{1T}\chi_T = \alpha_{3T} - \beta_{3T} = \alpha_{3T} - u_{a\theta}$, and rearranging, (S.28) becomes

$$\underbrace{\sigma_Y^2[u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T - \alpha_{3T}(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T)] + \psi_2\gamma_T\chi_T\alpha_{3T}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})^2}_{=:f(\alpha_{3T},\gamma_T,\chi_T)} = 0. \tag{S.31}$$

We construct C_ψ such that if $\psi \in (C_\psi/\gamma^o, 0]$, there exists a unique real α_{3T} continuous in (γ_T, χ_T) over $[0, \gamma^o] \times [0, 1]$ that solves (S.31) and has the same sign as $u_{a\theta}$; from this, we construct β_1 solving (S.28) and in turn, β_0 and β_2 solving (S.27) and (S.30), respectively, all continuously differentiable in (γ_T, χ_T).

If $\psi_2 = 0$, $f(\cdot, \gamma_T, \chi_T)$ is linear and has unique root $\alpha_{3T}(\gamma_T, \chi_T) := \frac{u_{a\theta}+u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T}{1-u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T}$, which is well-defined and has the same sign as $u_{a\theta}$ for all $(\gamma_T, \chi_T) \in [0, \gamma^o] \times [0, 1]$ by Assumption 2. And clearly, it is continuous in (γ_T, χ_T) over this domain.

Hence, for the remainder of the proof, assume $\psi_2 < 0$. We consider two cases: $\hat{u}_{a\hat{a}} = 0$ and $\hat{u}_{a\hat{a}} \neq 0$. If $\hat{u}_{a\hat{a}} = 0$, $f(\cdot, \gamma_T, \chi_T)$ is linear and has unique root $\alpha_{3T}(\gamma_T, \chi_T) := \frac{\sigma_Y^2(u_{a\theta}+u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T)}{\sigma_Y^2(1-u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T)-\psi_2\gamma_T\chi_T\hat{u}_{\hat{a}\theta}^2} = \frac{\sigma_Y^2(u_{a\theta}+u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T)}{\sigma_Y^2-\psi_2\gamma_T\chi_T\hat{u}_{\hat{a}\theta}^2}$ which is well-defined, has the same sign as $u_{a\theta}$, and is continuous in (γ_T, χ_T) over $[0, \gamma^o] \times [0, 1]$. Specifically, the numerator of the expression defining α_{3T} has the same sign as $u_{a\theta}$ by Assumption 2, and $\psi_2 \leq 0$ ensures that the denominator is positive. Thus, for $\hat{u}_{a\hat{a}} = 0$, the lemma holds with $C_\psi = -\infty$, provided that the remaining variables are uniquely determined and continuously differentiable; we perform this step for both cases $\hat{u}_{a\hat{a}} = 0$ and $\hat{u}_{a\hat{a}} \neq 0$ after solving for α_{3T} for the latter.

Now consider $\psi_2 < 0$ and $\hat{u}_{a\hat{a}} \neq 0$. If $\chi_T = 0$, then for all $\gamma_T \geq 0$, $f(\cdot, \gamma_T, \chi_T)$ is linear with intercept $\sigma_Y^2 u_{a\theta}$ and unique root $\alpha_{3T}(\gamma_T, 0) := u_{a\theta}$.

Next, suppose $\chi_T \in (0, 1]$, $\psi_2 < 0$, and $\hat{u}_{a\hat{a}} \neq 0$. We establish a condition such that for all $(\gamma_T, \chi_T) \in [0, \gamma^o] \times (0, 1]$, $f(\cdot, \gamma_T, \chi_T)$ is strictly decreasing, and thus it has exactly one real root. Clearly this holds for $\gamma_T = 0$. If $\gamma_T > 0$, then $f(\cdot, \gamma_T, \chi_T)$ is cubic, and it satisfies $\lim_{\alpha_{3T}\to+\infty} f(\alpha_{3T}, \gamma_T, \chi_T) = -\infty$ and $\lim_{\alpha_{3T}\to-\infty} f(\alpha_{3T}, \gamma_T, \chi_T) = +\infty$. We calculate

$$\frac{\partial}{\partial \alpha_{3T}} f(\alpha_{3T}, \gamma_T, \chi_T) = -\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T) + \psi_2\gamma_T\chi_T(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})(\hat{u}_{\hat{a}\theta} + 3\hat{u}_{a\hat{a}}\alpha_{3T}), \quad \text{(S.32)}$$

which is concave and quadratic in α_{3T}. The first term on the right hand side of (S.32) is negative by Assumption 2. The maximum value of the right hand side of (S.32), attained at $\alpha_{3T} = -\frac{2\hat{u}_{\hat{a}\theta}}{3\hat{u}_{a\hat{a}}}$, is $-\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\chi_T) - \frac{1}{3}\hat{u}_{\hat{a}\theta}^2\psi_2\gamma_T\chi_T$. Thus

$$\frac{\partial}{\partial \alpha_3} f(\alpha_{3T}, \gamma_T, \chi_T) \leq -\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\} - \frac{1}{3}\hat{u}_{\hat{a}\theta}^2\psi_2\gamma^o.$$

Define $C_\psi = -\infty$ if $\hat{u}_{\hat{a}\theta} = 0$ (or if $\hat{u}_{a\hat{a}} = 0$ as noted earlier) and $C_\psi := -\frac{3\sigma_Y^2 \min\{1,1-u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\hat{u}_{\hat{a}\theta}^2} < 0$ if $\hat{u}_{a\hat{a}}\hat{u}_{\hat{a}\theta} \neq 0$. By construction, for all $\psi \in (C_\psi/\gamma^o, 0]$ and all $(\gamma_T, \chi_T) \in [0, \gamma^o] \times (0, 1]$, $f(\cdot, \gamma_T, \chi_T)$ is strictly decreasing and has a unique real root which we denote $\alpha_{3T}(\gamma_T, \chi_T)$. Since $f(0, \gamma_T, \chi_T) = \sigma_Y^2[u_{a\theta} + u_{a\hat{a}}\hat{u}_{\hat{a}\theta}\chi_T]$ has the same sign as $u_{a\theta}$ by Assumption 2 and $f(\cdot, \gamma_T, \chi_T)$ is decreasing, $\alpha_{3T}(\gamma_T, \chi_T)$ has the same sign as $u_{a\theta}$.

Having characterized $\alpha_{3T}(\gamma_T, \chi_T) \neq 0$ on $[0, \gamma^o] \times [0, 1]$ for $(\psi_1, \psi_2) \in \mathbb{R} \times (C_\psi/\gamma^o, 0)$

for both cases $\hat{u}_{a\hat{a}} = 0$ and $\hat{u}_{a\hat{a}} \neq 0$, observe that $(\gamma_T, \chi_T) \mapsto \alpha_{3T}(\gamma_T, \chi_T)$ is continuously differentiable by the implicit function theorem.

It is immediate from (S.29) that in any solution, $\beta_{3T} = u_{a\theta}$; trivially, this is continuously differentiable in (γ_T, χ_T). Further, given $\alpha_{3T}(\gamma_T, \chi_T)$ as above, (S.28) uniquely determines $\beta_{1T}(\gamma_T, \chi_T)$ continuously differentiable. We now show there exist β_{0T} and β_{2T} solving (S.27) and (S.30), respectively, also continuously differentiable in (γ_T, χ_T). These equations are linear (and uncoupled) in β_{0T} and β_{2T}, respectively. Rearranging terms in (S.30) yields

$$\beta_{2T} \underbrace{\left[\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}) - \psi_2\gamma_T\alpha_{3T}\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})\right]}_{=:\tilde{C}(\gamma_T,\chi_T)}$$
$$= \hat{u}_{a\hat{a}}\beta_{1T}(1 - \chi_T)[\psi_2\gamma_T\alpha_{3T}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T}) + \sigma_Y^2 u_{a\hat{a}}].$$

Since collecting β_{0T} terms on the left side of (S.27) yields the same coefficient $\tilde{C}(\gamma_T, \chi_T)$, to establish existence it suffices to show that $\tilde{C}(\gamma_T, \chi_T) > 0$.

If $\alpha_{3T}\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T}) \geq 0$, we are done, since by Assumption 2, $\sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}) > 0$, and by assumption, $\psi_2 \leq 0$ and $\gamma_T \geq 0$. Suppose now that $\alpha_{3T}\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T}) < 0$. Note that this implies $\hat{u}_{a\hat{a}} \neq 0$ and $\hat{u}_{\hat{a}\theta} \neq 0$, and by the definition of C_ψ, $\psi_2 > C_\psi/\gamma^o = -\frac{3\sigma_Y^2 \min\{1, 1-u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\hat{u}_{\hat{a}\theta}^2\gamma^o}$. Thus, we have

$$\begin{aligned}
\tilde{C}(\gamma_T, \chi_T) &\geq \sigma_Y^2(1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}) - \left[-\frac{3\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\hat{u}_{\hat{a}\theta}^2\gamma^o}\right]\gamma_T\alpha_{3T}\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T}) \\
&\geq \sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\} + \left[\frac{3\sigma_Y^2}{\hat{u}_{\hat{a}\theta}^2} \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}\right]\alpha_{3T}\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T}) \\
&= \frac{\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\hat{u}_{\hat{a}\theta}^2}\left[\hat{u}_{\hat{a}\theta}^2 + 3\alpha_{3T}\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3T})\right] \\
&> \frac{\sigma_Y^2 \min\{1, 1 - u_{a\hat{a}}\hat{u}_{a\hat{a}}\}}{\hat{u}_{\hat{a}\theta}^2}\left[\hat{u}_{\hat{a}\theta} + \frac{3}{2}\alpha_{3T}\hat{u}_{a\hat{a}}\right]^2 \\
&\geq 0,
\end{aligned}$$

where the second line uses that $\gamma_T \leq \gamma^o$ and $\alpha_{3T}\hat{u}_{a\hat{a}}(\hat{u}_{\hat{a}\theta}+\hat{u}_{a\hat{a}}\alpha_{3T}) < 0$, and the fourth line uses that $\alpha_{3T}\hat{u}_{a\hat{a}} \neq 0$ which implies $(\alpha_{3T}\hat{u}_{a\hat{a}})^2 > 0$. Thus $\tilde{C}(\gamma_T, \chi_T) > 0$, so given α_{3T}, (S.27) and (S.30) have unique solutions β_{0T} and β_{2T} which by inspection are continuously differentiable over the domain $[0, \gamma^o] \times [0, 1]$. This concludes the proof of the lemma statement.

For later use in our existence theorem, we note the following facts about the solution described above. First, β_{2T} carries a factor of $1 - \chi_T$ and (therefore) v_{6T} carries $(1 - \chi_T)^2$, while $v_{8T} = 0$. Hence, it is easy to perform a change of variables $(\tilde{\beta}_{2T}, \tilde{v}_{6T}, \tilde{v}_{8T}) = (\beta_{2T}/(1 - \chi_T), \gamma_T v_{6T}/(1 - \chi_T)^2, \gamma_T v_{8T}/(1 - \chi_T))$ as in the main text, all continuously differentiable in

(γ_T, χ_T) and thus bounded over the compact domain $[0, \gamma^o] \times [0, 1]$. Second, after this change of variables, there exist nondecreasing functions $\eta, \bar{v} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\eta(\gamma^o), \bar{v}(\gamma^o) \to 0$ as $\gamma^o \to 0$ such that for all $(\gamma_T, \chi_T) \in [0, \gamma^o] \times [0, 1]$ and $i \in \{6, 8\}$, $||\vec{\beta}_T(\gamma_T, \chi_T) - \vec{\beta}^m(\chi_T)||_\infty \leq \eta(\gamma^o)$ and $|\tilde{v}_{iT}(\gamma_T, \chi_T)| \leq \bar{v}(\gamma^o)$. To see this, observe that the right hand sides of (S.27)-(S.30) converge uniformly to 0 as $\gamma_T \to 0$, and thus $\vec{\beta}_T(\gamma_T, \cdot)$ converges uniformly to $\vec{\beta}^m(\cdot)$ as $\gamma_T \to 0$. Similarly, it is easy to see that $\tilde{v}_{6T}(\gamma_T, \cdot) \to 0$ uniformly as $\gamma_T \to 0$, while $\tilde{v}_{8T}$ is identically zero. Setting $\eta(\gamma^o) := \sup\{||\vec{\beta}_T(\gamma_T, \chi_T) - \vec{\beta}^m(\chi_T)||_\infty : (\gamma_T, \chi_T) \in [0, \gamma^o] \times [0, 1]\}$ and $\bar{v}(\gamma^o) := \sup\{|\tilde{v}_{6T}(\gamma_T, \chi_T)| : (\gamma_T, \chi_T) \in [0, \gamma^o] \times [0, 1]\}$, we have that η and $\bar{v}$ are nondecreasing by construction, and they satisfy the inequalities and limit properties as claimed. □

S.3.3 Proof of Corollary C.2

We follow the same steps from before except for a few modifications, which we outline here. First, we note that the terminal values $(\beta_{1T}, \tilde{\beta}_{2T}, \beta_{3T}, \tilde{v}_{6T}, \tilde{v}_{8T})$ and $\alpha_{3T} = \beta_{1T}\chi_T + \beta_{3T} \neq 0$ are now implicit C^1 functions of (γ_T, χ_T) over $[0, \gamma^o] \times [0, 1]$ given by Lemma S.6.[2]

In the '**Centering**' step, we replace the initial conditions for the backward ODEs of $(\vec{\beta}^c, \tilde{v}_6, \tilde{v}_8)$ with the difference $(\beta_{1T}, \tilde{\beta}_{2T}, \beta_{3T}, \tilde{v}_{6T}, \tilde{v}_{8T}) - (\beta_1^m, \tilde{\beta}_2^m, \beta_3^m, 0, 0)$ (suppressing dependence on (γ_T, χ_T)), and observe that the ODEs themselves do not change. Likewise, in the '**Auxiliary variable**' step, we modify the initial condition for the (backward) $\tilde{\alpha}$-ODE to be $\tilde{\alpha}_0 = \alpha_{3T}(\gamma, \chi) \neq 0$ for all $(\gamma, \chi) \in [0, \gamma^o] \times [0, 1]$. Moreover, by the same comparison argument as before, $\tilde{\alpha}$ does not change sign; but since α_{3T} and α_3^m always have the same sign from Lemma S.6, it follows again that $\tilde{\alpha}/\alpha_3^m > 0$ from which we can find an interval of existence independent of $r \geq 0$. We also note that the argument showing that the solution to the boundary value problem satisfies $\alpha_3 = \tilde{\alpha} \neq 0$ also remains unchanged.

Step 1 of the proof of Theorem C.1 is only modified in that the parameter used in our domain $\Lambda(\cdot)$ will be $\rho + K + \eta(\gamma^o)$ instead of $\rho + K$, to account for nonzero initial conditions for the centered variables. We elaborate on this parameter when discussing Step 3 below.

In Step 2 of the proof of Theorem C.1, we write for $i \in \{1, 2, 3\}$ and $j \in \{4, 5\}$

$$|\mathbf{b}_{it} - \mathbf{b}_{i0}| = \left| \int_0^t e^{-r \int_s^t \frac{\tilde{\alpha}_u}{\alpha_{3u}^m} du} \hat{\gamma}_s h_i(\mathbf{b}_s - \mathbf{b}_0 + \mathbf{b}_0, \hat{\chi}_s) ds \right|$$

$$|\mathbf{b}_{jt} - \mathbf{b}_{j0}| = \left| \int_0^t e^{-\int_s^t (r + \hat{\gamma}_u R_j(\mathbf{b}_u, \hat{\chi}_u)) du} \hat{\gamma}_s h_j(\mathbf{b}_s - \mathbf{b}_0 + \mathbf{b}_0, \hat{\chi}_s) ds \right|.$$

Now from the end of the proof of Lemma S.6, there exist nondecreasing functions $\eta(\gamma^o)$

[2]Note that $\tilde{v}_{8T} = 0$ as before, but $\tilde{v}_{6T}$ can be nonzero.

and $\bar{v}(\gamma^o)$ with $\eta(\gamma^o), \bar{v}(\gamma^o) \to 0$ as $\gamma^o \to 0$ such that for all $i \in \{1,2,3\}$, $|\mathbf{b}_{i0}| \le \eta(\gamma^o)$ and for $j \in \{4,5\}$, $|\mathbf{b}_{j0}| \le \bar{v}(\gamma^o)$. Hence, when the bound $|\mathbf{b}_{is} - \mathbf{b}_{i0}| \le K$ holds for all $i \in \{1,\dots,5\}$, we can bound $|h_i(\mathbf{b}_s - \mathbf{b}_0 + \mathbf{b}_0, \chi_s)| \le h_i(K + \eta(\gamma^o), K + \bar{v}(\gamma^o))$ for scalars $h_i(K+\eta(\gamma^o), K+\bar{v}(\gamma^o))$ which are increasing in both arguments. Define $T(\gamma^o; K) := \min_{i\in\{1,\dots,5\}} \frac{K}{\gamma^o h_i(K+\eta(\gamma^o), K+\bar{v}(\gamma^o))}$. By repeating the arguments used in the proof of Lemma C.2, for all $T < T(\gamma^o; K)$, a solution to the modified version of $(\text{IVP}^{\text{bwd}}(\hat{\lambda}))$ exists and satisfies $|\mathbf{b}_{it} - \mathbf{b}_{i0}| \le K$ for all $t \in [0, T]$.

In Step 3, q is defined the same way as before, except now ω as in Lemma C.3 is the vector of new initial values for $(\vec{\beta}^c, \tilde{v}_6, \tilde{v}_8)$. Since ω is continuous, q remains continuous. Moreover,

$$\begin{aligned} q(\lambda) &= q(\lambda) - (\omega_1(\lambda_T), \omega_3(\lambda_T)) + (\omega_1(\lambda_T), \omega_3(\lambda_T)) \\ &= \underbrace{(\hat{\mathbf{b}}_1(\cdot;\lambda), \hat{\mathbf{b}}_3(\cdot;\lambda)) - (\omega_1(\lambda_T), \omega_3(\lambda_T))}_{||\cdot||_\infty \le K} + \underbrace{(\mathrm{B}_1(\lambda_{(\cdot)}), \mathrm{B}_3(\lambda_{(\cdot)}))}_{||\cdot||_\infty \le \rho} + \underbrace{(\omega_1(\lambda_T), \omega_3(\lambda_T))}_{||\cdot||_\infty \le \eta(\gamma^o)}, \end{aligned}$$

and thus the triangle inequality yields $||q(\lambda)||_\infty \le K + \rho + \eta(\gamma^o)$.

Step 4 goes through almost unchanged, except that $(\text{IVP}^{\text{fwd}}(q(\lambda)))$ now takes as its input $q(\lambda)$, bounded by $K + \rho + \eta(\gamma^o)$ as above. Applying this bound to $|\dot{\boldsymbol{\lambda}}_{1t}|$ and $|\dot{\boldsymbol{\lambda}}_{2t}|$, it follows that the solution $\boldsymbol{\lambda}$ to $(\text{IVP}^{\text{fwd}}(q(\lambda)))$ lies in $\Lambda(K+\rho+\eta(\gamma^o))$. By the same arguments as in the original Step 4, $q \mapsto \boldsymbol{\lambda}(q)$ is continuous, and the function g defined by $g(\lambda) = \boldsymbol{\lambda}(q(\lambda))$ is a continuous self-map on $\Lambda(K + \rho + \eta(\gamma^o))$. Schauder's Fixed Point Theorem then applies exactly as in Step 5. To conclude, we again define $T(\gamma^o)$ by maximizing $T(\gamma^o; K)$ over $K > 0$, and we note that $T(\gamma^o) \in \Omega(1/\gamma^o)$, and it can be written in the form C_T/γ^o.

S.3.4 Proof of Proposition 1: (β_0, v_3) System

When $u_{a\hat{a}} = \hat{u}_{a\hat{a}} = \psi_{a\hat{a}} = 0$, using that $\beta_1 = \beta_2 = v_6 = v_8 = 0$, the ODEs for β_0 and v_3 are

$$\dot{\beta}_{0t} = r\frac{\alpha_{3t}}{u_{a\theta}}(\beta_{0t} - u_{a0}) - \frac{v_{3t}\alpha_{3t}(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3t})^2\gamma_t^2\chi_t}{\sigma_X^2\sigma_Y^2(1-\chi_t)} - \frac{\alpha_{3t}\gamma_t(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3t})(u_{\hat{a}\theta}(\beta_{0t} - u_{a0}) + u_{\hat{a}0}u_{a\theta})}{u_{a\theta}\sigma_Y^2}$$

$$\dot{v}_{3t} = rv_{3t} + \frac{v_{3t}\gamma_t\chi_t(\hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3t})^2}{1-\chi_t}.$$

The terminal condition for v_3 is $v_{3T} = 0$, so $v_3 = 0$ is the unique solution. It is easy to see that after setting $v_{3t} = 0$ in the ODE for β_{0t}, there is no dependence on σ_X and χ.

S.4 Extension: Both Players Affect X and Y

In this section, we generalize several results from the baseline model to allow both players' actions to (additively) enter both signals:

$$dX_t = (\hat{a}_t + \nu a_t)dt + \sigma_X dZ_t^X, \qquad \nu \in [0,1] \tag{S.33}$$

$$dY_t = (a_t + \hat{\nu}\hat{a}_t)dt + \sigma_Y dZ_t^Y, \qquad \hat{\nu} \in [0,1]. \tag{S.34}$$

Subsection S.4.1 extends our results from Section 3 in the paper. First, Lemma S.7 establishes a generalized version of our representation result (Lemma 1 in the paper). Second, Lemma S.8, stated for the more general signal structure above, verifies that the public state corresponds to the belief about θ using only the public information. Third, Lemma S.9 presents laws of motion for (M, L) for arbitrary strategies for our sender. These results enable us to set up a best-response problem for our sender that is analogous to the one from Section 3.3 in the paper. Subsequently, Lemma S.10 establishes that $0 < \gamma \leq \gamma^o$ and $0 \leq \chi < 1$ (in particular, confirming that the law of motion of L is always well-defined), and that the ODE system for (γ, χ) admits a unique solution (which confirms that the (γ, χ) delivered by our fixed-point approach indeed corresponds to the variances of the players' learning). We conclude this part by showing a one-to-one mapping between χ and γ for this general case (Proposition S.3), analogous to Proposition 9 in the paper.

Finally, in Subsection S.4.2 we leverage these results and our fixed-point method to provide an existence result analogous to Theorem 1 but for our trading game from Section 4.3. Subsection S.4.3 contains the proof of Proposition 7 in the paper.

S.4.1 Technical Results

Lemma S.7. *Suppose that (X, Y) is driven by (S.33)–(S.34) and the receiver believes that (8)—i.e., $M_t = \chi_t\theta + (1-\chi_t)L_t$, with χ a deterministic function—holds, where $(L_t)_{t\in[0,T]}$ is a process that depends only on the public information.*[3] *Then (8) holds at all times if and only if $L_t = \mathbb{E}[\theta|\mathcal{F}_t^X]$ under (S.33)–(S.34), $\chi_t = \mathbb{E}_t[(M_t - \hat{M}_t)^2]/\gamma_t$, and*

$$\begin{aligned}
\dot{\gamma}_t &= -\gamma_t^2(\beta_{3t} + \beta_{1t}\chi_t)^2\Sigma, & \gamma_0 &= \gamma^o, & \text{(S.35)}\\
\dot{\chi}_t &= \gamma_t(\beta_{3t} + \beta_{1t}\chi_t)^2\Sigma(1-\chi_t) - \gamma_t(\nu[\beta_{3t} + \beta_{1t}\chi_t] + \delta_{1t}\chi_t)^2/\sigma_X^2, & \chi_0 &= 0, & \text{(S.36)}\\
dL_t &= (l_{0t} + l_{1t}L_t)dt + B_t dX_t, & L_0 &= \mu, & \text{(S.37)}
\end{aligned}$$

where $\Sigma := \nu^2/\sigma_X^2 + 1/\sigma_Y^2$ and (l_{0t}, l_{1t}, B_t) are deterministic and given in (S.44)-(S.46).

[3] Formally, $(L_t)_{t\in[0,T]}$ can be any square-integrable process progressively measurable w.r.t. $(\mathcal{F}_t^X)_{t\in[0,T]}$.

Proof. Let L in (8) denote a square-integrable process that is progressively measurable with respect to $(\mathcal{F}_t^X)_{t\in[0,T]}$. Inserting (8) into (6) yields $a_t = \alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}\theta$ which the receiver thinks drives X and Y, where $\alpha_{0t} = \beta_{0t}$, $\alpha_{2t} = \beta_{2t} + \beta_{1t}(1-\chi_t)$, and $\alpha_{3t} = \beta_{3t} + \beta_{1t}\chi_t$.

The receiver's filtering problem is then conditionally Gaussian. Specifically, define

$$\begin{aligned} d\hat{X}_t &:= dX_t - [\hat{a}_t + \nu(\alpha_{0t} + \alpha_{2t}L_t)]dt = \nu\alpha_{3t}\theta dt + \sigma_X dZ_t^X \\ d\hat{Y}_t &:= dY_t - [\alpha_{0t} + \alpha_{2t}L_t + \hat{\nu}\hat{a}_t]dt = \alpha_{3t}\theta dt + \sigma_Y dZ_t^Y, \end{aligned}$$

which are in the receiver's information set, and where the last equalities hold from his perspective. By Theorems 12.6 and 12.7 in Liptser and Shiryaev (1977), his posterior belief is Gaussian with mean $\hat{M}_t$ and variance γ_{1t} (simply γ_t in the main body) that evolve as

$$d\hat{M}_t = \frac{\nu\alpha_{3t}\gamma_{1t}}{\sigma_X^2}[d\hat{X}_t - \nu\alpha_{3t}\hat{M}_t dt] + \frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y^2}[d\hat{Y}_t - \alpha_{3t}\hat{M}_t dt] \text{ and } \dot{\gamma_{1t}} = -\gamma_{1t}^2\alpha_{3t}^2\Sigma, \tag{S.38}$$

with $\Sigma := \nu^2/\sigma_X^2 + 1/\sigma_Y^2$. (These expressions still hold after deviations, which go undetected.)

The sender can affect $\hat{M}_t$ via her choice of actions. Indeed, using that $d\hat{X}_t = \nu(a_t - \alpha_{0t} - \alpha_{2t}L_t)dt + \sigma_X dZ_t^X$ and $d\hat{Y}_t = (a_t - \alpha_{0t} - \alpha_{2t}L_t)dt + \sigma_Y dZ_t^Y$ from her standpoint,

$$d\hat{M}_t = (\kappa_{0t} + \kappa_{1t}a_t + \kappa_{2t}\hat{M}_t)dt + B_t^X dZ_t^X + B_t^Y dZ_t^Y, \text{ where} \tag{S.39}$$

$$\kappa_{1t} = \alpha_{3t}\gamma_{1t}\Sigma, \quad \kappa_{0t} = -\kappa_{1t}[\alpha_{0t} + \alpha_{2t}L_t],\ \kappa_{2t} = -\alpha_{3t}\kappa_{1t},\ B_t^X = \frac{\nu\alpha_{3t}\gamma_{1t}}{\sigma_X},\ B_t^Y = \frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y}. \tag{S.40}$$

On the other hand, since the sender always thinks that the receiver is on path, the public signal evolves, from her perspective, as $dX_t = (\nu a_t + \delta_{0t} + \delta_{1t}\hat{M}_t dt + \delta_{2t}L_t)dt + \sigma_X dZ_t^X$. Because the dynamics of $\hat{M}$ and X have drifts that are affine in $\hat{M}$—with intercepts and slopes that are in the sender's information set—and deterministic volatilities, the pair $(\hat{M}, X)$ is conditionally Gaussian. Thus, by the filtering equations in Theorem 12.7 in Liptser and Shiryaev (1977), $M_t := \mathbb{E}_t[\hat{M}_t]$ and $\gamma_{2t} := \mathbb{E}_t[(M_t - \hat{M}_t)^2]$ satisfy

$$\begin{aligned} dM_t &= \underbrace{(\kappa_{0t} + \kappa_{1t}a_t + \kappa_{2t}M_t)dt}_{=\mathbb{E}_t[(\kappa_{0t}+\kappa_{1t}a_t+\kappa_{2t}\hat{M}_t)dt]} + \frac{\sigma_X B_t^X + \gamma_{2t}\delta_{1t}}{\sigma_X^2}[dX_t - (\nu a_t + \delta_{0t} + \delta_{1t}M_t + \delta_{2t}L_t)dt] \quad \text{(S.41)} \\ \dot{\gamma}_{2t} &= 2\kappa_{2t}\gamma_{2t} + (B_t^X)^2 + (B_t^Y)^2 - \left(B_t^X + \gamma_{2t}\delta_{1t}/\sigma_X\right)^2, \end{aligned}$$

with $dZ_t := [dX_t - (\nu a_t + \delta_{0t} + \delta_{1t}M_t + \delta_{2t}L_t)dt]/\sigma_X$ a Brownian motion from the sender's standpoint.[4] Critically, observe that since (S.41) is linear, one can solve for M_t as an *explicit*

[4]Theorem 12.7 in Liptser and Shiryaev (1977) is stated for actions that depend on (θ, X) exclusively, but it also applies to those that condition on past play (i.e., on M). Indeed, from (S.39), $\hat{M}_t = \hat{M}_t^\dagger + A_t$ where $M_t^\dagger = M_t^\dagger[Z_s^X, Z_t^Y; s < t]$ and $A_t = \int_0^t e^{\int_0^s \kappa_{2u}du}\kappa_{1s}a_s ds$. Applying the theorem to $(\hat{M}_t^\dagger, X_t - \int_0^t \nu a_s ds)_{t\in[0,T]}$,

function of past actions $(a_s)_{s<t}$ and past realizations of the public history $(X_s)_{s<t}$.

Inserting $a_t = \beta_{0t} + \beta_{1t}M_t + \beta_{2t}L_t + \beta_{3t}\theta$ in (S.41) and collecting terms yields

$$\begin{aligned}
dM_t &= [\hat{\kappa}_{0t} + \hat{\kappa}_{1t}M_t + \hat{\kappa}_{2t}L_t + \hat{\kappa}_{3t}\theta]dt + \hat{B}_t dX_t, \qquad \text{where} \\
\hat{\kappa}_{0t} &= \alpha_{3t}\gamma_{1t}\Sigma(\beta_{0t} - \alpha_{0t}) - \frac{\nu\alpha_{3t}\gamma_{1t} + \gamma_{2t}\delta_{1t}}{\sigma_X^2}[\nu\beta_{0t} + \delta_{0t}] \\
\hat{\kappa}_{1t} &= \alpha_{3t}\gamma_{1t}\Sigma(\beta_{1t} - \alpha_{3t}) - \frac{\nu\alpha_{3t}\gamma_{1t} + \gamma_{2t}\delta_{1t}}{\sigma_X^2}[\nu\beta_{1t} + \delta_{1t}] \\
\hat{\kappa}_{2t} &= \alpha_{3t}\gamma_{1t}\Sigma(\beta_{2t} - \alpha_{2t}) - \frac{\nu\alpha_{3t}\gamma_{1t} + \gamma_{2t}\delta_{1t}}{\sigma_X^2}[\nu\beta_{2t} + \delta_{2t}] \\
\hat{\kappa}_{3t} &= \left[\frac{\alpha_{3t}\gamma_{1t}}{\sigma_Y^2} - \frac{\nu\gamma_{2t}\delta_{1t}}{\sigma_X^2}\right]\beta_{3t}, \qquad \hat{B}_t = \frac{\nu\alpha_{3t}\gamma_{1t} + \gamma_{2t}\delta_{1t}}{\sigma_X^2}
\end{aligned}$$

Let $R(t,s) = \exp(\int_s^t \hat{\kappa}_{1u}du)$. Since $M_0 = \mu$, we have

$$M_t = R(t,0)\mu + \theta\int_0^t R(t,s)\hat{\kappa}_{3s}ds + \int_0^t R(t,s)[\hat{\kappa}_{0s} + \hat{\kappa}_{2s}L_s]ds + \int_0^t R(t,s)\hat{B}_s dX_s.$$

As in the main body, imposing equality with (8) yields the equations

$$\chi_t = \int_0^t R(t,s)\hat{\kappa}_{3s}ds \text{ and } L_t = \frac{R(t,0)\mu + \int_0^t R(t,s)[\hat{\kappa}_{0s} + \hat{\kappa}_{2s}L_s]ds + \int_0^t R(t,s)\hat{B}_s dX_s}{1-\chi_t}.$$

The validity of the construction boils down to finding a solution to the previously stated equation for χ that takes values in $[0,1)$. Indeed, when this is the case, it is easy to see that

$$dL_t = \{L_t[\hat{\kappa}_{1t} + \hat{\kappa}_{2t} + \hat{\kappa}_{3t}]dt + \hat{\kappa}_{0t}dt + \hat{B}_t dX_t\}/(1-\chi_t), \tag{S.42}$$

from which it is easy to conclude that L is a (linear) function of X as conjectured.

We will find a solution to the χ-equation that is C^1 with values in $[0,1)$. Differentiating $\chi_t = \int_0^t R(t,s)\hat{\kappa}_{3s}ds$ then yields an ODE for χ as below that is coupled with γ_1 and γ_2:

$$\begin{aligned}
\dot{\gamma}_{1t} &= -\gamma_{1t}^2(\beta_{3t} + \beta_{1t}\chi_t)^2\Sigma \\
\dot{\gamma}_{2t} &= -2\gamma_{2t}\gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_t)^2\Sigma + \gamma_{1t}^2(\beta_{3t} + \beta_{1t}\chi_t)^2\Sigma \\
&\quad - (\nu\gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_t) + \gamma_{2t}\delta_{1t})^2/\sigma_X^2 \\
\dot{\chi}_t &= \gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_t)^2\Sigma(1-\chi_t) \\
&\quad - (\nu[\beta_{3t} + \beta_{1t}\chi_t] + \delta_{1t}\chi_t)(\nu\gamma_{1t}(\beta_{3t} + \beta_{1t}\chi_t) + \gamma_{2t}\delta_{1t})/\sigma_X^2.
\end{aligned}$$

yields a posterior mean $M_t^\dagger$ and variance $\gamma_{2t}^\dagger$ for $\hat{M}_t^\dagger$ such that $M_t^\dagger + A_t = M_t$ as in (S.41) and $\gamma_{2t} = \gamma_{2t}^\dagger$.

The identity $\chi = \gamma_2/\gamma_1 \in [0, 1)$ can be established by the same steps as in the baseline model. Setting $\gamma_2 = \chi\gamma_1$ in the third ODE, and writing γ for γ_1, the first and third ODEs become (S.35)–(S.36). Using the expressions that define $(\vec{\kappa}, \hat{B})$ yields that (S.42) becomes

$$dL_t = (\ell_{0t} + \ell_{1t}L_t)dt + B_t dX_t, \tag{S.43}$$

where

$$l_{0t} = -\frac{\gamma_t(\nu\beta_{0t} + \delta_{0t})(\nu\alpha_{3t} + \chi_t\delta_{1t})}{\sigma_X^2(1-\chi_t)} \tag{S.44}$$

$$l_{1t} = -\frac{\gamma_t[\nu(\beta_{1t} + \beta_{2t} + \beta_{3t}) + \delta_{1t} + \delta_{2t}](\nu\alpha_{3t} + \chi_t\delta_{1t})}{\sigma_X^2(1-\chi_t)} \tag{S.45}$$

$$B_t = \frac{\nu\alpha_{3t}\gamma_t + \gamma_t\chi_t\delta_{1t}}{\sigma_X^2(1-\chi_t)}. \tag{S.46}$$

That L_t coincides with $\mathbb{E}[\theta|\mathcal{F}_t^X]$ is proved in Lemma S.8 below. Note that by setting $\nu = 0$ in (S.43)-(S.46), we recover the law of motion for L, (A.10), in the main paper. □

Lemma S.8 (State L as a Public Belief)**.** *The process L is the belief about θ held by an outsider who observes only X. Moreover,* $\begin{pmatrix}\theta\\ \hat{M}_t\end{pmatrix}|\mathcal{F}_t^X \sim \mathcal{N}(M_t^{out}, \gamma_t^{out})$ *where* $M_t^{out} = \begin{pmatrix}L_t\\ L_t\end{pmatrix}$ *and* $\gamma_t^{out} = \begin{pmatrix}\frac{\gamma_t}{1-\chi_t} & \frac{\gamma_t\chi_t}{1-\chi_t}\\ \frac{\gamma_t\chi_t}{1-\chi_t} & \frac{\gamma_t\chi_t}{1-\chi_t}\end{pmatrix}$.

Proof. The outsider jointly filters the state $v_t = (\theta, \hat{M}_t)'$. For the evolution of the state and the signal, we adopt notation from Section 12.3 in Liptser and Shiryaev (1977). From the outsider's perspective, both players (and in particular player 2) are on the equilibrium path, and thus the outsider believes that v_t evolves as

$$dv_t = a_1(t, X)v_t dt + b_1(t, X)dW_1(t) + b_2(t, X)dW_2(t),$$

where $a_1(t, X) := \begin{pmatrix}0 & 0\\ \alpha_{3t}^2\gamma_t\Sigma & -\alpha_{3t}^2\gamma_t\Sigma\end{pmatrix}$, $b_1(t, X) := \begin{pmatrix}0 & 0\\ 0 & \frac{\alpha_{3t}\gamma_t}{\sigma_Y}\end{pmatrix}$, $b_2(t, X) := \begin{pmatrix}0\\ \frac{\nu\alpha_{3t}\gamma_t}{\sigma_X}\end{pmatrix}$, $W_1(t) := \begin{pmatrix}W_{11}(t)\\ Z_t^Y\end{pmatrix}$ and $W_2(t) := Z_t^X$, where $W_{11}(t)$ is a standard Brownian motion and $W_{11}(t), Z_t^Y$ and Z_t^X are mutually independent. The signal is

$$dX_t^{out} := dX_t - [\delta_{0t} + \delta_{2t}L_t + \nu(\alpha_{0t} + \alpha_{2t}L_t)]dt = A_1(t, X)v_t + B_1(t, X)W_1(t) + B_2(t, X)W_2(t),$$

where $A_1(t, X) := \begin{pmatrix}\nu\alpha_{3t} & \delta_{1t}\end{pmatrix}$, $B_1(t, X) := \begin{pmatrix}0 & 0\end{pmatrix}$ and $B_2(t, X) = \sigma_X$.

Hence, denoting $M_t^{out} = \begin{pmatrix} M_{t,1}^{out} \\ M_{t,2}^{out} \end{pmatrix}$ and $\gamma_t^{out} = \begin{pmatrix} \gamma_{t,11}^{out} & \gamma_{t,12}^{out} \\ \gamma_{t,21}^{out} & \gamma_{t,22}^{out} \end{pmatrix}$ and imposing $\gamma_{t,21}^{out} = \gamma_{t,12}^{out}$, we have from the standard filtering equations of Liptser and Shiryaev (1977, Theorem 12.7) that $\begin{pmatrix} \theta \\ \hat{M}_t \end{pmatrix} |\mathcal{F}_t^X \sim \mathcal{N}(M_t^{out}, \gamma_t^{out})$, where M^{out} and γ^{out} are the unique solutions to

$$dM_t^{out} = a_1(t,X)M_t^{out} + \frac{1}{\sigma_X^2}\left[\begin{pmatrix} 0 \\ \nu\alpha_{3t}\gamma_t \end{pmatrix} + \gamma_t^{out}\begin{pmatrix} \nu\alpha_{3t} \\ \delta_{1t} \end{pmatrix}\right]\left\{dX_t^{out} - (\nu\alpha_{3t}M_{t,1}^{out} + \delta_{1t}M_{t,2}^{out})dt\right\} \tag{S.47}$$

$$\begin{aligned} \dot{\gamma}_t^{out} &= a_1(t,X)\gamma_t^{out} + \gamma_t^{out}a_1^* + \begin{pmatrix} 0 & 0 \\ 0 & \Sigma^2\gamma_t^2\alpha_{3t}^2 \end{pmatrix} \\ &\quad - \frac{1}{\sigma_X^2}\left[\begin{pmatrix} 0 \\ \nu\alpha_{3t}\gamma_t \end{pmatrix} + \gamma_t^{out}\begin{pmatrix} \nu\alpha_{3t} \\ \delta_{1t} \end{pmatrix}\right]\left[\begin{pmatrix} 0 \\ \nu\alpha_{3t}\gamma_t \end{pmatrix} + \gamma_t^{out}\begin{pmatrix} \nu\alpha_{3t} \\ \delta_{1t} \end{pmatrix}\right]^* \end{aligned}$$

with initial conditions $M_0^{out} = \begin{pmatrix} \mu \\ \mu \end{pmatrix}$ and $\gamma_0^{out} = \begin{pmatrix} \gamma^o & 0 \\ 0 & 0 \end{pmatrix}$.

Recall that γ_1 and χ satisfy $\dot{\gamma}_{1t} = -\alpha_{3t}^2\gamma_t^2\Sigma$ and $\dot{\chi}_t = \gamma_t\alpha_{3t}^2\Sigma(1-\chi_t) - \gamma_t(\nu\alpha_{3t} + \delta_{1t}\chi_t)^2/\sigma_X^2$ with initial conditions $\gamma_{1,0} = \gamma^o$ and $\chi_0 = 0$. It is straightforward to verify that $\gamma_t^{out} = \begin{pmatrix} \frac{\gamma_t}{1-\chi_t} & \frac{\gamma_t\chi_t}{1-\chi_t} \\ \frac{\gamma_t\chi_t}{1-\chi_t} & \frac{\gamma_t\chi_t}{1-\chi_t} \end{pmatrix}$ satisfies the γ^{out}-ODE above along with given initial condition. Moreover, γ_t^{out} is positive semidefinite as its leading principal minors are positive multiples of 1 and $\chi - \chi^2 > 0$.

Next, substitute given the solution γ_t^{out} into (S.47) and subtract the equation for the second component from its first to obtain the following SDE for $\bar{M} := M_1^{out} - M_2^{out}$: $d\bar{M}_t = -\Sigma\bar{M}_t\alpha_{3t}^2\gamma_t$ with initial condition $\bar{M}_0 = 0$. Now if $\bar{M}_t > 0$, then $d\bar{M}_t < 0$, giving us a contradiction; likewise for the case $\bar{M}_t < 0$. It follows that $\bar{M}_t = 0$, and thus $M_{t,1}^{out} = M_{t,2}^{out}$, for all $t \geq 0$. Substituting this back into (S.47), we have

$$\begin{aligned} dM_{t,1}^{out} &= \frac{\gamma_t(\nu\alpha_{3t} + \delta_{1t}\chi_t)}{\sigma_X^2(1-\chi_t)}(dX_t^{out} - (\nu\alpha_{3t} + \delta_{1t})M_{t,1}^{out}dt) \\ &= \frac{\gamma_t(\nu\alpha_{3t} + \delta_{1t}\chi_t)}{\sigma_X^2(1-\chi_t)}[dX_t - (\nu\alpha_{0t} + \delta_{0t} + M_{1,t}^{out}(\nu\alpha_{3t} + \delta_{1t}) + L_t(\nu\alpha_{2t} + \delta_{2t}))dt]. \end{aligned}$$

On the other hand, we have

$$dL_t = \frac{L_t[\hat{\kappa}_{1t} + \hat{\kappa}_{2t} + \hat{\kappa}_{3t}]dt + \hat{\kappa}_{0t}dt + \hat{B}_t dX_t}{1-\chi_t}$$

$$= \frac{\gamma_t(\nu\alpha_{3t} + \delta_{1t}\chi_t)}{\sigma_X^2(1-\chi_t)} [dX_t - (\delta_{0t} + \nu\alpha_{0t} + L_t[\nu(\alpha_{2t} + \alpha_{3t}) + \delta_{1t} + \delta_{2t}])dt] \,.$$

Hence $\bar{L}_t := M_{t,1}^{out} - L_t$ satisfies $d\bar{L}_t = -\frac{\gamma_t(\nu\alpha_{3t}+\delta_{1t}\chi_t)}{\sigma_X^2(1-\chi_t)}\bar{L}_t(\nu\alpha_{3t} + \delta_{1t})$ with initial condition $\bar{L}_0 = \mu - \mu = 0$. We conclude that $\bar{L}_t = 0$, and thus $L_t = M_{t,1}^{out} = M_{t,2}^{out}$, for all $t \geq 0$. □

Lemma S.9 (Sender's controlled dynamics)**.** *Suppose that the receiver follows (7) and believes that (6) and (8) hold. Then, if the sender follows* $(a_t')_{t\in[0,T]}$*,*

$$dM_t = \gamma_t\alpha_{3t}\left(\frac{\nu^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2}\right)(a_t' - [\alpha_{0t} + \alpha_{2t}L_t + \alpha_{3t}M_t])dt + \frac{\nu\alpha_{3t}\gamma_t + \chi_t\gamma_t\delta_{1t}}{\sigma_X}dZ_t \tag{S.48}$$

$$dL_t = \frac{\gamma_t(\nu\alpha_{3t} + \delta_{1t}\chi_t)}{\sigma_X^2(1-\chi_t)}\{[\nu(a_t' - [\alpha_{0t} + (\alpha_{2t} + \alpha_{3t})L_t]) + \delta_{1t}(M_t - L_t)]\,dt + \sigma_X dZ_t\}\,, \tag{S.49}$$

where $Z_t := \frac{1}{\sigma_X}[X_t - \int_0^t(\nu a_s' + \delta_{0s} + \delta_{1s}M_s + \delta_{2s}L_s)ds]$ *is a Brownian motion from the sender's perspective. Also,* $\mathbb{E}_t[(M_t - \hat{M}_t)^2] = \gamma_t\chi_t$ *for any such* $(a_t')_{t\in[0,T]}$*.*

Proof. Equation (S.48) follows from using (S.40) in (S.41), and (S.49) follows from (S.37) using (S.44)-(S.46) and that $dX_t = (\nu a_t + \delta_{0t} + \delta_{2t}L_t + \delta_{1t}M_t)dt + \sigma_X dZ_t$ from the sender's perspective. □

Lemma S.10 (Learning ODEs)**.** *Suppose that* $(\beta_1, \beta_3, \delta_1)$ *are differentiable. Then, there is a unique solution to* (S.35)–(S.36)*, and this solution satisfies* $0 < \gamma_t \leq \gamma^o$ *and* $0 \leq \chi_t < 1$ *for all* $t \in [0,T]$*, with strict inequalities over* $(0,T]$ *if* $\beta_{3,0} \neq 0$*. The same conclusions hold if* $\delta_{1t} = \hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3t}$*. Moreover, in both cases,* $\chi_t \leq 1 - \gamma_t/\gamma^o$ *for all* $t \in [0,T]$*.*

Proof. The same arguments from the baseline proof go through, so we focus on proving the tighter inequality $\chi_t \leq 1 - \gamma_t/\gamma^o$.

We now use the comparison theorem. Let $f^\chi(t,\chi_t)$ denote the right hand side of (S.36). Note that $z_t = 1 - \gamma_t/\gamma^o$ solves the ODE

$$\dot{z}_t = f^z(t,z_t) := \gamma_t(\beta_{1t}\chi_t + \beta_{3t})^2\Sigma(1-z_t)$$

with $z_0 = 0$. Thus $\chi_0 = z_0$. Further, note that $f^z(t,z_t) \geq f^\chi(t,z_t)$, and hence

$$0 = \dot{\chi}_t - f^\chi(t,\chi_t) = \dot{z}_t - f^z(t,z_t) \leq \dot{z}_t - f^\chi(t,z_t).$$

By the comparison theorem, $\chi_t \leq z_t$ for all $t \in [0,T]$, as desired. The same argument applies when $\delta_{1t} = \hat{u}_{\hat{a}\theta} + \hat{u}_{a\hat{a}}\alpha_{3t}$ in (S.36). □

The following result generalizes the one-to-one mapping between γ and χ in Proposition 9 to the case where the sender's action enters the public signal, as in (S.33).

Proposition S.3 (One-to-one mapping)**.** *Suppose that signals have the form* (S.33)-(S.34)*, and suppose that* $\hat{u}_{\hat{a}\theta} = 0$*. Then* $\chi_t = \frac{c_1 c_2(1-[\gamma_t/\gamma^o]^d)}{c_1+c_2[\gamma_t/\gamma^o]^d}$ *for some positive scalars* c_1, c_2 *and* d*. Thus,* $\chi_t \in [0, c_2)$ *when* $\gamma_t \in (0, \gamma^o]$*.*

Proof. We first derive a candidate mapping for the general case of a drift $\hat{a}_t + \nu a_t$, $\nu \in [0,1]$, in X. Suppose $\delta_1 = \hat{u}_{a\hat{a}}\alpha_3$. The χ-ODE for $\nu \in [0,1]$ boils down to

$$\dot{\chi}_t = \gamma_t \alpha_{3t}^2 \left(\left[\frac{\nu^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2} \right] (1-\chi_t) - \frac{(\nu + \hat{u}_{a\hat{a}}\chi_t)^2}{\sigma_X^2} \right) =: -\gamma_t \alpha_{3t}^2 Q(\chi_t).$$

If $f : [0, \bar{\chi}) \to [0, \gamma^o]$, some $\bar{\chi} \in (0,1]$, is differentiable and $f(\chi_t) = \gamma_t$ for all $t \geq 0$, then $f'(\chi_t)\dot{\chi}_t = \dot{\gamma}_t$. When $\alpha_{3t} \neq 0$, $\frac{f'(\chi_t)}{f(\chi_t)} = \frac{\Sigma}{Q(\chi_t)}$. Hence, we solve the ODE $\frac{f'(\chi)}{f(\chi)} = \frac{\Sigma}{Q(\chi)}$ for $\chi \in (0, \bar{\chi})$ where $f(0) = \gamma^o$.

To this end, let $c_2 := \frac{\sqrt{b^2+4(\hat{u}_{a\hat{a}})^2/[\sigma_X\sigma_Y]^2}-b}{2(\hat{u}_{a\hat{a}}/\sigma_X)^2}$ and $-c_1 := \frac{-\sqrt{b^2+4(\hat{u}_{a\hat{a}})^2/[\sigma_X\sigma_Y]^2}-b}{2(\hat{u}_{a\hat{a}}/\sigma_X)^2}$, where $b := [\nu^2/\sigma_X^2 + 1/\sigma_Y^2] + 2\nu\hat{u}_{a\hat{a}}/\sigma_X^2$, be the roots of the convex quadratic Q above. Note that these are well-defined since $\hat{u}_{\hat{a}\theta}$ and Assumption 1 part (ii) imply that $\hat{u}_{a\hat{a}} \neq 0$.

Clearly, $-c_1 < 0 < c_2$. Also, $c_2 \leq 1$ as $Q(1) \geq 0$. Thus, $\frac{\Sigma}{Q(\chi)} = -\frac{\sigma_X^2\Sigma}{(\hat{u}_{a\hat{a}})^2(c_1+c_2)} \left[\frac{1}{\chi+c_1} - \frac{1}{\chi-c_2} \right]$ is well defined (and negative) over $[0, c_2)$ with $1/(\chi+c_1) > 0$ and $-1/(\chi-c_2) > 0$ over the same domain. We can then set $\bar{\chi} = c_2$ and solve $\int_0^\chi \frac{f'(s)}{f(s)} ds = -\frac{\sigma_X^2\Sigma}{(\hat{u}_{a\hat{a}})^2(c_1+c_2)} \log\left(\frac{\chi+c_1}{c_2-\chi} \frac{c_2}{c_1} \right)$, which yields the decreasing function $f(\chi) = f(0) \left(\frac{c_1}{c_2} \right)^{1/d} \left(\frac{c_2-\chi}{\chi+c_1} \right)^{1/d}$, where

$$1/d := \sigma_X^2 \Sigma / [(\hat{u}_{a\hat{a}})^2 (c_1 + c_2)] > 0.$$

Imposing $f(0) = \gamma^o$ and inverting yields $\chi(\gamma) = f^{-1}(\gamma)$ as given in the lemma. Note that $\chi(\gamma^o) = 0$ and $\chi(0) = c_2$. Since χ is decreasing, we have $\chi_t \in [0, c_2)$ when $\gamma_t \in (0, \gamma^o]$. Finally, routine calculations akin to those in the proof of Proposition 9 confirm that this function satisfies the χ-ODE for $\nu \neq 0$. □

S.4.2 Trading Game (Section 4.3): Existence Result

The following result adapts Theorem 1 to our trading game.

Proposition S.4. *There exists a scalar* $C > 0$ *independent of* γ^o *such that if* $T < C/\gamma^o$*, there exists an LME.*

Proof of Proposition S.4. The proof has the same structure as that for Theorem 1. Below we describe the main variations—we refer the reader to that central proof for all the details. We characterize a LME in which the sender trades according to $a_t = \beta_{3t}(\theta - L_t) + \beta_{1t}(M_t - L_t)$

and the sender's value function has the form

$$V(\theta, m, \ell, t) = v_{0t} + v_{At}(\theta - \ell)^2 + v_{Bt}(\theta - m)^2 + v_{Ct}(m - \ell)^2,$$

where (v_0, v_A, v_B, v_C) are C^1 functions of time. Clearly, the receiver's strategy must be $\hat{a}_t = \delta_{1t}(\hat{M}_t - L_t)$, where $\delta_{1t} := 1$ for all $t \in [0, T]$. The laws of motion for $\hat{M}, M$, and L are given by (S.38), (S.48), and (S.49), respectively.

By imposing the first order condition on the right hand side of the HJB equation and matching coefficients, we obtain identities allowing us to solve for (v_B, v_C) in terms of $(\gamma, \chi, \beta_1, \beta_3, v_A)$.

Using these identities (and their derivatives) in the first order condition, we obtain a core subsystem of ODEs in $(\gamma, \chi, \beta_1, \beta_3, v_A)$. We use a change of variables $\tilde{v}_A = v_A\gamma$. Since the myopic coefficients $(\beta^m_{1t}, \beta^m_{3t}) = (0, 1)$ are constant over time, there is no need to work with centered coefficients. Going forward in time, the ODEs for $(\beta_1, \beta_3, \tilde{v}_A)$ have the form

$$\begin{aligned}\dot{x}_t &= \frac{\gamma_t h_x(\beta_{1t}, \beta_{3t}, \tilde{v}_{At}, \chi_t)}{\sigma_X^2\sigma_Y^2(1-\chi_t)^2 j(\beta_{1t}, \beta_{3t}, \tilde{v}_{At}, \chi_t)}, \qquad x \in \{\beta_1, \beta_3\},\\ \dot{\tilde{v}}_{At} &= \gamma_t\left\{\tilde{v}_{At}\frac{h_{v,1}(\beta_{1t}, \beta_{3t}, \chi_t)}{\sigma_X^2\sigma_Y^2(1-\chi_t)} + h_{v,2}(\beta_{1t}, \beta_{3t}, \chi_t)\right\},\end{aligned}$$

where $h_x, j, h_{v,1}, h_{v,2}$ are polynomials for $x \in \{\beta_1, \beta_3\}$. The full expressions for these ODEs can be found in the Mathematica file `spm.nb` on our websites.

Of particular interest for our bounding exercise is the denominator term j, which involves more terms than in the baseline model and reads

$$\begin{aligned}j(\beta_{1t}, \beta_{3t}, \tilde{v}_{At}, \chi_t) = \sigma_X^4\alpha_{3t} - \chi_t\sigma_X^2\alpha_{3t}(\sigma_X^2 + \sigma_Y^2[1+\chi_t] + 2\tilde{v}_{At})]\\ + \sigma_Y^2\chi_t[\chi_t(2\tilde{v}_{At} + \sigma_X^2(1 - \beta_{1t}[1-\chi_t])) - \sigma_X^2],\end{aligned} \tag{S.50}$$

where $\alpha_{3t} = \beta_{1t}\chi_t + \beta_{3t}$. The terminal conditions are $(\beta_{1T}, \beta_{3T}, \tilde{v}_{AT}) = (0, 1, 0)$.

Also going forward the ODEs for (γ, χ) are

$$\dot{\gamma}_t = -\gamma_t^2(\beta_{3t} + \beta_{1t}\chi_t)^2\Sigma \tag{S.51}$$

$$\dot{\chi}_t = \gamma_t(\beta_{3t} + \beta_{1t}\chi_t)^2\Sigma(1-\chi_t) - \gamma_t(\nu[\beta_{3t} + \beta_{1t}\chi_t] + \delta_{1t}\chi_t)^2/\sigma_X^2, \tag{S.52}$$

where $\nu = 1$ and $\delta_{1t} = 1$, subject to $(\gamma_0, \chi_0) = (\gamma^o, 0)$.

These ODEs and initial/terminal conditions define a BVP in $\mathbf{z} := (\gamma, \chi, \beta_1, \beta_3, \tilde{v}_A)$ that we write as $\dot{\mathbf{z}}_t = F(\mathbf{z}_t)$.

Proposition S.5. *There exists $C > 0$ independent of γ^o such that there exists a solution to* *the BVP whenever $T < C/\gamma^o$.*

Proof. We follow the steps of the original proof, modifying them as needed.

Step 1: *Define the domain for our fixed point equation.* Given $K \in (0,1)$, we define $\Gamma(K)$ as the space of uniformly Lipschitz continuous functions $\gamma : [0,T] \to [0,\gamma^o]$ with uniform Lipschitz constant $(\gamma^o)^2(2K+1)^2/\sigma_Y^2$. For fixed $K \in (0,1)$ and fixed $\bar{\chi} \in (0,1)$ we define $X(K,\bar{\chi})$ as the space of Lipschitz continuous functions $\chi : [0,T] \to [0,\bar{\chi}]$ with uniform Lipschitz constant $\gamma^o\,[(2K+1)^2\Sigma + (2K+2)^2/\sigma_X^2]$.

We desire to choose K and $\overline{\chi}(K)$ so that the RHS of the ODEs for β_1 and β_3 are well-defined for $(\beta_1, \beta_3, \tilde{v}_A, \chi) \in \mathcal{B}(K) \times [0, \overline{\chi}(K)]$, where $\mathcal{B}(K) := [-K,K] \times [1-K, 1+K] \times [-K,K]$; in particular, we choose $\overline{\chi}$ to ensure that the polynomial j in the denominator of $\dot{\beta}_{1t}$ and $\dot{\beta}_{3t}$ is bounded away from zero. Recalling (S.50), it is easy to see that for any $K \in (0,1)$, there exists $\overline{\chi}(K) \in (0,1)$ such that j is positive and bounded away from zero over the domain $\mathcal{B}(K) \times [0, \overline{\chi}(K)]$. Of course, since on this domain $\chi_t \le \overline{\chi}(K) < 1$, the term $(1-\chi_t)$ on the right hand sides of $(\dot{\beta}_{1t}, \dot{\beta}_{3t}, \dot{\tilde{v}}_{At})$ is positive and bounded away from zero.

In some abuse of notation, define $X(K) = X(K, \overline{\chi}(K))$ and $\Lambda(K) := \Gamma(K) \times X(K)$.

Step 2: *Given $\lambda = (\gamma, \chi) \in \Lambda(K)$, define a backward initial value problem (IVP) for $(\beta_1, \beta_3, \tilde{v}_A)$, and establish sufficient conditions for this IVP to have a unique solution.* Recall that the hat notation reverses time. We define the backward IVP

$$\dot{\mathbf{b}}_t = \mathrm{f}^{\hat{\lambda}}(\mathbf{b}_t, t) \;\; \text{s.t.} \;\; \mathbf{b}_0 = (0,1,0). \tag{IVP$^{\text{bwd}}(\hat{\lambda})$}$$

We argue that there exists a positive threshold $T^{\text{bwd}}(\gamma^o; K) \in \Omega(1/\gamma^o)$ such that for all $\lambda \in \Lambda(K)$, a unique solution $\mathbf{b}(\cdot;\lambda)$ to (IVP$^{\text{bwd}}(\hat{\lambda})$) exists over $[0,T]$ and satisfies $\mathbf{b} \in \mathcal{B}(K)$. By standard results, a local solution exists, and solutions are unique given existence. Now if over some interval of existence we have $\mathbf{b}_t \in \mathcal{B}(K)$ for $i \in \{1,2,3\}$, then we also have

$$|\mathbf{b}_{it} - \mathbf{b}_{i0}| \le \left| \int_0^t \dot{\mathbf{b}}_{it}\, dt \right| \le \int_0^t \gamma^o h_i(K)\, dt = t\gamma^o h_i(K),$$

where $h_i(K)$ is a positive scalar that bounds the magnitude of the right hand side the associated ODE using $\mathbf{b}_t \in \mathcal{B}(K)$ and $\chi_t \in [0, \overline{\chi}(K)]$, where the latter holds by the definition of our domain $\Lambda(K)$. Now for all $K \in (0,1)$, define $T^{\text{bwd}}(\gamma^o; K) = \min_{i \in \{1,2,3\}} \frac{K}{\gamma^o h_i(K)} > 0$. Clearly, $T^{\text{bwd}}(\gamma^o; K) \in \Omega(1/\gamma^o)$. By an analogous argument to the one in the baseline proof, this construction implies that for all $T < T^{\text{bwd}}(\gamma^o; K)$ and $\lambda \in \Lambda(K)$, (IVP$^{\text{bwd}}(\hat{\lambda})$) has a unique solution over $[0,T]$ satisfying $\mathbf{b}_t \in \mathcal{B}(K)$ for all $t \in [0,T]$. We denote this solution by

$\mathbf{b}(\cdot;\lambda)$, and we define the functional

$$q(\lambda) := (\hat{\mathbf{b}}_1(\cdot;\lambda), \hat{\mathbf{b}}_2(\cdot;\lambda))$$

mapping λ to *forward*-oriented solutions for β_1 and β_3, to be used in the forward-evolving learning ODEs (S.51)-(S.52).

Step 3: *The operator $\lambda \mapsto q(\lambda)$ is continuous and $||q(\lambda) - (0,1)||_\infty < K$ for all $\lambda \in \Lambda(K)$.* Continuity follows from Lemma C.3 and the bound follows from Step 2.

Step 4: *Construct a continuous self-map on $\Lambda(K)$ using the IVP for the learning ODEs.* As before, for all $\lambda \in \Lambda(K)$, we define an IVP for $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$,

$$\dot{\boldsymbol{\lambda}}_t = \mathrm{f}^{q(\lambda)}(\boldsymbol{\lambda}_t, t) \ \text{ s.t. } \ \boldsymbol{\lambda}_0 = (\gamma^o, 0), \tag{IVP$^{\text{fwd}}(q(\lambda))$}$$

consisting of the two forward learning ODEs (S.51)-(S.52), where $(q_1(\lambda), q_2(\lambda))$ play the role of (β_1, β_3).

There is a unique solution $\boldsymbol{\lambda}(q(\lambda))$ over $[0,T]$ to (IVP$^{\text{fwd}}(q(\lambda))$) by Lemma S.10 and it satisfies $\boldsymbol{\lambda}_t \in [0,\gamma^o] \times [0,1]$ for all time. The arguments that this solution satisfies the Lipschitz bounds defining $\Lambda(K)$ are the same as before.

However, there is one extra step to ensure that $\boldsymbol{\lambda}(q(\lambda)) \in \Lambda(K)$: we must verify that $\boldsymbol{\lambda}_2 \in [0, \overline{\chi}(K)]$. To that end, note that the solution (γ, χ) to the system (S.51)-(S.52) with initial conditions $(\gamma_0, \chi_0) = (\gamma^o, 0)$ satisfies $\chi_t \leq 1 - \gamma_t/\gamma^o$ for all $t \in [0,T]$ by Lemma S.10, so $\boldsymbol{\lambda}_2(q(\lambda)) \leq 1 - \boldsymbol{\lambda}_1(q(\lambda))/\gamma^o$. In turn, given a bound $|\beta_{3t} + \beta_{1t}\chi_t| \leq A$, it is easy to show that $\gamma_t \geq \frac{\gamma^o}{T\gamma^o A^2 \Sigma + 1}$. As $|q_1(\lambda)|, |q_2(\lambda)| \leq K$ and $\boldsymbol{\lambda}_2 \in [0,1]$, we have $|q_2(\lambda) + q_1(\lambda)\boldsymbol{\lambda}_2(q(\lambda))| \leq 2K+1$, so letting $2K+1$ play the role of A, we have $\boldsymbol{\lambda}_1(q(\lambda)) \geq \frac{\gamma^o}{T\gamma^o(2K+1)^2\Sigma + 1}$. Combining these inequalities,

$$\boldsymbol{\lambda}_2(q(\lambda)) \leq 1 - \boldsymbol{\lambda}_1(q(\lambda))/\gamma^o \leq 1 - \frac{1}{T\gamma^o(2K+1)^2\Sigma + 1} = \frac{T\gamma^o(2K+1)^2\Sigma}{T\gamma^o(2K+1)^2\Sigma + 1}. \tag{S.53}$$

There exists a threshold $T^{\text{fwd}}(\gamma^o; K) \in \Omega(1/\gamma^o)$ such that $T < T^{\text{fwd}}(\gamma^o; K)$ implies the last upper bound in (S.53) is less than $\overline{\chi}(K)$.

Thus, if we define $T(\gamma^o; K) := \min\{T^{\text{fwd}}(\gamma^o; K), T^{\text{bwd}}(\gamma^o; K)\}$, $T < T(\gamma^o; K)$ implies that $g(\lambda) := \boldsymbol{\lambda}(q(\lambda))$ is a self-map on $\Lambda(K)$, and it is continuous by the same reasoning as before. We conclude by optimizing over $K \in (0,1)$, defining $T(\gamma^o) := \max_{K \in (0,1)} T(\gamma^o; K)$ and finally $C := T(\gamma^o)\gamma^o$.[5] □

[5] Note that as $K \uparrow 1$, $\overline{\chi}(K) \downarrow 0$, so $T^{\text{fwd}}(\gamma^o; K) \downarrow 0$. And as $K \downarrow 0$, $T^{\text{bwd}}(\gamma^o; K) \downarrow 0$. Hence, $T(\gamma^o; K)$ is maximized at interior K.

Having proven Proposition S.5, it is straightforward to recover the value function coefficients (v_B, v_C) from the identities used earlier, and the ODE for v_0 has a unique solution since it is linear as usual, concluding the proof. □

S.4.3 Trading Game (Section 4.3): Proof of Proposition 7

Proof of Proposition 7. To prove that $\beta_{0t} = 0$, $\beta_{1t} + \beta_{2t} + \beta_{3t} = 0$ in any LME, we follow similar steps to those used in the proof of Proposition 2. Using the general form of the conjectured value function in Section 5 and deriving the ODEs from the sender's best response problem (see `spm.nb`), it is easy to verify that $(\beta_0, v_3) = (0, 0)$ jointly solve their respective ODEs and the terminal conditions $(\beta_{0T}, v_{3T}) = 0$ in any LME. By the Picard-Lindelöf theorem, these are the unique solutions. Next, we show that $S_t := \beta_{1t} + \beta_{2t} + \beta_{3t} = 0$ for all t. From the terminal conditions $\beta_{1T} = 0$, $\beta_{2T} = -1$, and $\beta_{3T} = 1$, we have $S_T = 0$.

Define the following candidate solution for v_8:

$$v_{8t}^{cand} = \frac{\sigma_Y^2[-\sigma_X^2 + \sigma_X^2\beta_{1t}(1-\chi_t)^2 + \chi_t(\sigma_X^2 + 2v_{6t}\gamma_t)]}{(\sigma_X^2+\sigma_Y^2)\alpha_{3t}\gamma_t(1-\chi_t)} + \frac{\alpha_{3t}[\sigma_X^2\sigma_Y^2(1-\chi_t) + 2v_{6t}\gamma_t(-\sigma_X^2[1-\chi_t] + \sigma_Y^2\chi_t)]}{(\sigma_X^2+\sigma_Y^2)\alpha_{3t}\gamma_t(1-\chi_t)},$$

which satisfies $v_{8T}^{cand} = 0 = v_{8T}$. Define $v_{8t}^{\Delta} = v_{8t}^{cand} - v_{8t}$, which has terminal value $v_{8T}^{\Delta} = 0$.

We can construct a pair of ODEs for S_t and v_{8t}^{Δ} by differentiating each and using $\beta_{2t} = S_t - (\beta_{1t} + \beta_{3t})$ to eliminate β_{2t} and $v_{8t} = v_{8t}^{\Delta} - v_{8t}^{cand}$ to eliminate v_{8t} in each.

Routine calculation then shows that $(S_t, v_{8t}^{\Delta}) = (0, 0)$ solve the ODEs for S_t and v_{8t}^{Δ} and their terminal conditions. By Picard-Lindelöf, this is the unique solution, so in particular, $\beta_{1t}+\beta_{2t}+\beta_{3t} = 0$ in any solution. Hence $a_t = \beta_{1t}M_t+\beta_{2t}L_t+\beta_{3t}\theta = \beta_{3t}(\theta-L_t)+\beta_{1t}(M_t-L_t)$. Since $\alpha_{3T} = \beta_{1T}\chi_T+\beta_{3T} = 1 > 0$, and the right hand side of the ODE for α_3 contains a factor of α_3, a standard comparison theorem argument establishes that $\alpha_{3t} > 0$ for all $t \in [0, T]$.

On the path of play, using the representation $M_t = \chi_t\theta + (1-\chi_t)L_t$, we have $a_t = \beta_{3t}(\theta-L_t)+\beta_{1t}(M_t-L_t) = \beta_{3t}(\theta-L_t)+\beta_{1t}([\chi_t\theta+(1-\chi_t)L_t]-L_t) = \alpha_{3t}(\theta-L_t)$. Substituting the equilibrium strategies into (S.49) yields $dL_t = \Lambda_t dX_t$, where $\Lambda_t = \frac{\gamma_t^X(\alpha_{3t}+\chi_t)}{\sigma_X^2}$. □

S.4.4 Trading Game (Section 4.3): Footnotes 31 and 33

Proofs for Footnote 31 We now show that $\beta_{1t} > 0$, $\beta_{3t} \in (0, 1)$, and $\beta_{2t} < 0$; while $\frac{d\alpha_{3t}}{dt} > 0$, $\frac{d\beta_{3t}}{dt} > 0$, and $\frac{d\beta_{1t}}{dt} < 0$. We begin with some preliminary observations. Fix any $K \in (0, 1)$ and note that the right hand sides of the ODEs for $(\gamma, \chi, \beta_1, \beta_3, \tilde{v}_A)$ are of class C^1 in these variables on $[0, \gamma^o] \times [0, \overline{\chi}(K)] \times \mathcal{B}(K)$, where $\overline{\chi}(K)$ and $\mathcal{B}(K)$ were defined in the proof of Proposition S.5. Further, note that the proof of Proposition S.5 shows that for any $K \in (0,1)$, there exists $T(\gamma^o; K)$ such that for all $T < T(\gamma^o; K)$, a solution to the BVP exists, and in turn a LME exists, for which $\chi_t \leq \overline{\chi}(K)$ and $(\beta_{1t}, \beta_{3t}, \tilde{v}_{At}) \in \mathcal{B}(K)$ for all $t \in [0,T]$. (And recall $\gamma_t \in (0, \gamma^o]$ and $\chi_t \geq 0$ from Lemma S.10.) Finally, recall that $T(\gamma^o; K) \to 0$ as $K \to 0$, and thus $\chi \to 0$ uniformly by the upper bound (S.53). Hence, as $K \to 0$, we have $\chi \to 0$, $\beta_1 \to 0$, $\beta_3 \to 1$, and $\alpha \to 1$ uniformly, and in particular, $\beta_3 > 0$.

Evaluating the right hand sides of $(\dot{\beta}_{1t}, \dot{\beta}_{3t}, \dot{\alpha}_{3t})$ at $(\chi_t, \beta_{1t}, \beta_{3t}, \alpha_{3t}) = (0,0,1,1)$ yields $\left(-\frac{\alpha_{3t}(1-\alpha_{3t})\gamma_t}{\sigma_X^2}, \frac{\alpha_{3t}^3\gamma_t}{\sigma_X^2}, \frac{\alpha_{3t}^3\gamma_t}{\sigma_X^2}\right)$. Further, for all sufficiently small K, all $T < T(\gamma^o; K)$, and $t \in [0,T]$, $\frac{\alpha_{3t}^3\gamma_t}{\sigma_X^2}$ is positive and bounded away from zero. Hence, for sufficiently small K and any $T < T(\gamma^o; K)$, we have that β_3 and α_3 are strictly increasing.

We now show that $\dot{\beta}_1 < 0$; given that its terminal value is zero, this implies $\beta_1 > 0$, and given $\beta_3 > 0$, this in turn implies $\beta_2 < 0$. Since $\dot{\beta}_1$ converges to 0 uniformly, to sign $\dot{\beta}_1$, we examine the second derivative, $\ddot{\beta}_1$. It is easy to show that for small K, $\dot{\beta}_{1t} = 0$ implies $|\ddot{\beta}_{1t} - \alpha_{3t}^3\gamma_t^2/\sigma_X^4| < \epsilon(K)$, where $\epsilon(K) \to 0$ as $K \to 0$. Since $\alpha_{3t}^3\gamma_t^2/\sigma_X^4$ is strictly positive and bounded away from zero, this shows that for sufficiently small K, $\dot{\beta}_{1t} = 0$ implies $\ddot{\beta}_{1t} > 0$. Further, it is easy to check that in any LME (with no constraints on time horizon) $\dot{\beta}_{1T} = 0$. Hence, $\dot{\beta}_{1t}$ can only cross zero from below: we have $\dot{\beta}_{1t} < 0$ for all $t \in [0,T)$.

Proof or Footnote 33 The following proposition formalizes the claim made in footnote 33. We use superscript *leak* to denote $\sigma_Y \in (0, +\infty)$ and *no leak* to denote $\sigma_Y = +\infty$

Proposition S.6. *Fix $\sigma_Y \in (0, +\infty)$, and suppose that a LME exists over $[0,T]$. For any such LME, there exists a nonzero measure of times t for which $\Lambda_t > \Lambda_t^{no\ leak}$.*

Proof. First, note that $\alpha_{3T} = 1$ for all $\sigma_Y \in (0, +\infty) \cup \{+\infty\}$, while $\chi_T^{leak} > 0$ and $\chi_T^{no\ leak} = 0$. Thus $\alpha_{3T}^{leak} + \chi_T^{leak} > \alpha_{3T}^{no\ leak} + \chi_T^{no\ leak}$. We prove the result in each of two cases.

Case (i): $\gamma_T^{X,leak} \geq \gamma_T^{X,no\ leak}$. Since $\alpha_{3T}^{leak} + \chi_T^{leak} > \alpha_{3T}^{no\ leak} + \chi_T^{no\ leak}$, we have $\Lambda_T^{leak} > \Lambda_T^{no\ leak}$ unambiguously. By continuity, $\Lambda_t^{leak} > \Lambda_t^{no\ leak}$ for all t in a neighborhood of T.

Case (ii): $\gamma_T^{X,leak} < \gamma_T^{X,no\ leak}$. Here, since the initial values coincide, $\gamma_0^{X,leak} = \gamma_0^{X,no\ leak} = \gamma^o$, we must have $\dot{\gamma}_t^{X,leak} < \dot{\gamma}_t^{X,no\ leak}$ for a nonzero measure of times. It is easy to verify that $\dot{\gamma}_t^X = -\sigma_X^2\Lambda_t^2$, so it follows that $\Lambda_t^{leak} > \Lambda_t^{no\ leak}$ for a nonzero measure of times. □

Figure 1 below compares equilibrium strategies, signaling coefficients, and price impact for a myopic receiver, as in our baseline model, and a forward-looking receiver with discount rate $\hat{r} = 0$ (like the sender). The plots illustrate that the qualitative properties of our equilibrium are preserved (except for the obvious change in the receiver's signaling coefficient δ_1). Relative to a patient receiver, a myopic receiver trades more aggressively (Figure 1c), as he ignores the price impact of his own trades. Because of this, the sender's trades have more price impact, so the sender trades less aggressively on her private information θ if the receiver is myopic: β_3 falls as in Figure 1b. However, Figure 1a shows that β_1 moves in the opposite direction: if the sender has traded more in the past, the expects stronger upward drift in the price in the presence of a myopic receiver, and to arbitrage this, she trades more aggressively on the second-order private information. Figure 1d shows the effect on the total signaling coefficient is small since β_1 and δ_1 move in the opposite direction that β_3 moves. Lastly, Figure 1e shows that with a myopic receiver, price impact is: initially lower (due to the sender's lower β_3); then rises more quickly (due to the receiver's contribution to the total signaling coefficient); and finally, it falls faster as T approaches due to the market maker's learning, because a myopic receiver does not speed up his trades enough by the end. Here, "single agent" refers to the case where there is no receiver, only the sender and market maker.

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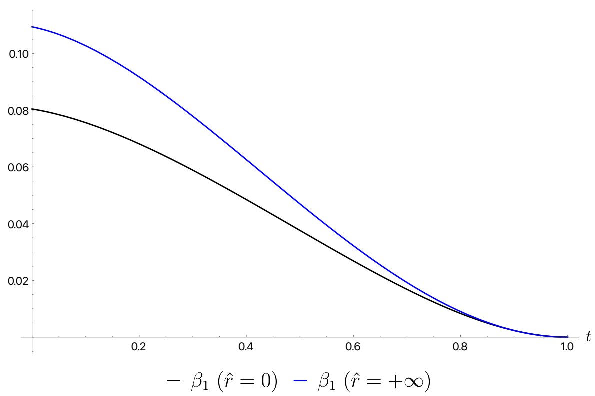
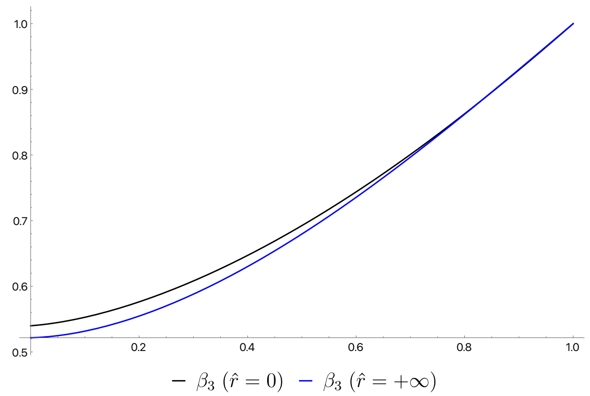
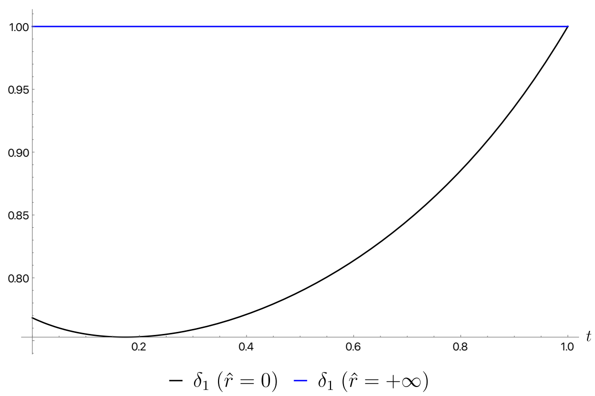


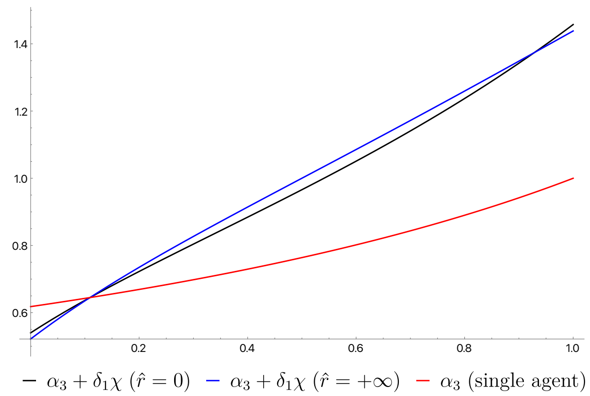
(a) Sender's β_1 coefficient



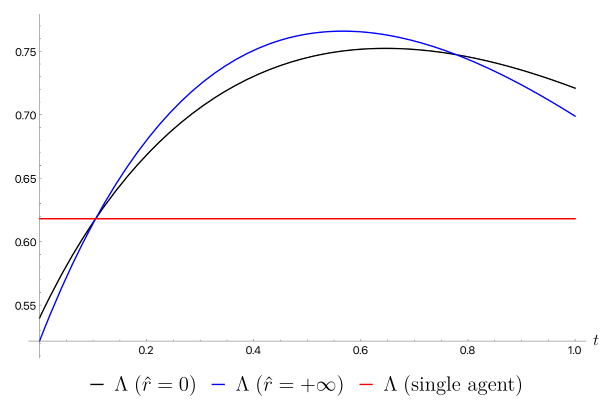
(b) Sender's β_3 coefficient



(c) Receiver's δ_1 coefficient



(d) Total signaling coefficient across players



(e) Price impact

Figure 1: Trading game: $(\gamma^o, r, \sigma_X) = (1, 0, 1)$ and $\sigma_Y = 0.5$ unless otherwise specified.